ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

Handout 16 Principles of Digital Communications
Midterm exam Apr. 21, 2023

4 problems, 41 points, 165 minutes.
1 sheet (2 pages) of notes allowed.

Good Luck!

PLEASE WRITE YOUR NAME ON EACH SHEET OF YOUR ANSWERS.

PLEASE WRITE THE SOLUTION OF EACH PROBLEM ON A SEPARATE SHEET.

PROBLEM 1. (10 points)
Consider a hypothesis testing problem where the hypothesis H can take the values 0 or 1 with equal probability. The observation $Y = (Y_1, Y_2)$, when $H = i$, is given by

$$Y = c_i + Z,$$

where $c_0 = (1, 2)$ and $c_1 = (2, 1)$ are vectors in $\mathbb{R}^2$ and Z is Gaussian, zero mean, with covariance matrix $K = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.
Let α be a real number and let $T = (1-\alpha)Y_1 + \alpha Y_2$ be a 1-dimensional statistic.

(a) (2 pts) What is the probability distribution of the statistic T when $H = i$, $i = 0, 1$?

(b) (2 pts) Consider the MAP decision rule *based only on the statistic* T. What is the error probability for this rule?

(c) (3 pts) Which choice of α will minimize the error probability in (b)?

(d) (3 pts) Is the statistic T, with the α of (c), a sufficient statistic?

PROBLEM 2. (10 points)
Consider a communication channel with input (x_1, x_2) in $\mathbb{R}^2$, and output (Y_1, Y_2) in $\mathbb{R}^2$ given by

$$Y_1 = A_1 x_1 - A_2 x_2 + Z_1$$
$$Y_2 = A_2 x_1 + A_1 x_2 + Z_2$$

where A_1, A_2, Z_1, Z_2 are all i.i.d. $\mathcal{N}(0, 1)$ random variables.

(a) (2 pts) Observe that (Y_1, Y_2) is a Gaussian vector for any given (x_1, x_2). Find its mean and covariance matrix in terms of x_1, x_2.

(b) (3 pts) Suppose $c_1, \ldots, c_m$ are m vectors in $\mathbb{R}^2$, and that when the message H equals i, the vector c_i is input to the communication channel above. The receiver, from the observation (Y_1, Y_2) tries to guess the value of H. Show that, no matter how the vectors $c_1, \ldots, c_m$ are chosen, $T = Y_1^2 + Y_2^2$ is a sufficient statistic.

(c) (2 pts) Suppose $m = 4$, and $c_1 = (5, 0), c_2 = (0, 5), c_3 = (3, 4), c_4 = (4, 3)$. All four messages are equally likely. What is the probability of error of the MAP decoder?

(d) (3 pts) Consider four designs, all with $m = 2$, all with equally likely messages:

1. $c_1 = (0, 0), c_2 = (10, 0)$
2. $c_1 = (8, 6), c_2 = (0, 0)$
3. $c_1 = (0, 0), c_2 = (5, 0)$
4. $c_1 = (8, 6), c_2 = (10, 0)$

With P_1, P_2, P_3, P_4 denoting the error probability with MAP decoding of these systems, how will $P_1, \ldots, P_4$ be ordered? What is the value of P_4?

PROBLEM 3. (11 points)
Consider a communication system over an additive white Gaussian noise channel (with noise intensity = 1) with two equally likely messages transmitted via waveforms $w_0(t) = -w_1(t) = \sqrt{\mathcal{E}}\mathbb{1}\{|t| < \frac{1}{2}\}$.

(a) (3 pts) At the receiver suppose we pass the received signal $R(t)$ through a filter with impulse response $h(t) = \mathbb{1}\{|t| < \frac{1}{2}\}$, sample the filter output at $t_0 = 0$, and, with Y denoting the value of the sample, decide $\hat{H} = 1$ if $Y < 0$, $\hat{H} = 0$ if $Y \geq 0$. Is this receiver optimal? What is the probability of error?

We are asked to design a receiver who does not get to observe $R(t)$ but observes the output $S(t)$ of a filter whose input is $R(t)$ and whose impulse response is $\mathbb{1}\{|t| < 1\}$. We can sample $S(t)$ at any number of time instants $t_1, \ldots, t_n$, and base our decision on the values $Y_1 = S(t_1), \ldots, Y_n = S(t_n)$.

(b) (2 pts) Suppose we choose $n = 1$, and $t_1 = 0$. What is the distribution of Y_1 given $H = i$, $i = 0, 1$?

(c) (3 pts) What is the optimal choice of $\hat{H}(Y_1)$ and what is the corresponding probability of error?

(d) (3 pts) Suppose we choose $n = 2$, with $t_1 = -1/2, t_2 = 1/2$. What is the optimal choice of $\hat{H}(Y_1, Y_2)$ and what is the probability of error?
Hint: $Y_1 + Y_2$ is a sufficient statistic.

PROBLEM 4. (10 points)
Consider a communication system for the AWGN channel with noise intensity $N_0/2$ with four equally likely messages, and suppose the waveforms $w_1, \dots, w_4$ have all unit norm and that $\langle w_i, w_k \rangle = \alpha$ for all $i \neq k$.

(a) (2 pts) Express $\|w_1 + w_2 + w_3 + w_4\|^2$ in terms of α, and show that $-1/3 \leq \alpha \leq 1$.

(b) (3 pts) Let $\{\tilde{w}_1, \dots, \tilde{w}_4\}$ be obtained by a translation of $\{w_1, \dots, w_4\}$ so that the new signal set is of minimal average energy. Do $\tilde{w}_1, \dots, \tilde{w}_4$ all have the same energy? If so, what is this energy in terms of α?

(c) (2 pts) Is there a common value of $\langle \tilde{w}_i, \tilde{w}_k \rangle$ for $i \neq k$? If so, what is the common value in terms of α?

(d) (3 pts) Let c_1, c_2, c_3, c_4 be the the corners of a regular tetrahedron in $\mathbb{R}^3$, centered at the origin. I.e., (i) $\|c_i\|^2 = A^2$ for all i, (ii) $\langle c_i, c_k \rangle = -A^2/3$ for all $i \neq k$. (As a consequence, $c_1 + c_2 + c_3 + c_4 = 0$.)

Let $e_{\text{tetra}}(A)$ denote the error probability of the MAP decoder that observes $Y = c_i + Z$ where Z is $\mathcal{N}(0, I_3)$ where each of the four c_i's are equally likely. Express the probability of error of the communication system (the system which uses the waveforms w_1, w_2, w_3, w_4) described at the start of the problem in terms of α, N_0 and $e_{\text{tetra}}(\cdot)$.

Hint: No lengthy computations are needed.