Handout 16
Midterm exam

Principles of Digital Communications

4 problems, 41 points, 165 minutes.
1 sheet (2 pages) of notes allowed.

Good Luck!

Please write your name on each sheet of your answers.

Please write the solution of each problem on a separate sheet.
Problem 1. (10 points)
Consider a hypothesis testing problem where the hypothesis $H$ can take the values 0 or 1 with equal probability. The observation $Y = (Y_1, Y_2)$, when $H = i$, is given by

$$Y = c_i + Z,$$

where $c_0 = (1, 2)$ and $c_1 = (2, 1)$ are vectors in $\mathbb{R}^2$ and $Z$ is Gaussian, zero mean, with covariance matrix $K = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

Let $\alpha$ be a real number and let $T = (1 - \alpha)Y_1 + \alpha Y_2$ be a 1-dimensional statistic.

(a) (2 pts) What is the probability distribution of the statistic $T$ when $H = i$, $i = 0, 1$?

(b) (2 pts) Consider the MAP decision rule based only on the statistic $T$. What is the error probability for this rule?

(c) (3 pts) Which choice of $\alpha$ will minimize the error probability in (b)?

(d) (3 pts) Is the statistic $T$, with the $\alpha$ of (c), a sufficient statistic?
Problem 2. (10 points)
Consider a communication channel with input \((x_1, x_2)\) in \(\mathbb{R}^2\), and output \((Y_1, Y_2)\) in \(\mathbb{R}^2\) given by
\[
Y_1 = A_1 x_1 - A_2 x_2 + Z_1 \\
Y_2 = A_2 x_1 + A_1 x_2 + Z_2
\]
where \(A_1, A_2, Z_1, Z_2\) are all i.i.d. \(\mathcal{N}(0, 1)\) random variables.

(a) (2 pts) Observe that \((Y_1, Y_2)\) is a Gaussian vector for any given \((x_1, x_2)\). Find its mean and covariance matrix in terms of \(x_1, x_2\).

(b) (3 pts) Suppose \(c_1, \ldots, c_m\) are \(m\) vectors in \(\mathbb{R}^2\), and that when the message \(H\) equals \(i\), the vector \(c_i\) is input to the communication channel above. The receiver, from the observation \((Y_1, Y_2)\) tries to guess the value of \(H\). Show that, no matter how the vectors \(c_1, \ldots, c_m\) are chosen, \(T = Y_1^2 + Y_2^2\) is a sufficient statistic.

(c) (2 pts) Suppose \(m = 4\), and \(c_1 = (5, 0), c_2 = (0, 5), c_3 = (3, 4), c_4 = (4, 3)\). All four messages are equally likely. What is the probability of error of the MAP decoder?

(d) (3 pts) Consider four designs, all with \(m = 2\), all with equally likely messages:
1. \(c_1 = (0, 0), c_2 = (10, 0)\)
2. \(c_1 = (8, 6), c_2 = (0, 0)\)
3. \(c_1 = (0, 0), c_2 = (5, 0)\)
4. \(c_1 = (8, 6), c_2 = (10, 0)\)

With \(P_1, P_2, P_3, P_4\) denoting the error probability with MAP decoding of these systems, how will \(P_1, \ldots, P_4\) be ordered? What is the value of \(P_4\)?
Problem 3. (11 points)
Consider a communication system over an additive white Gaussian noise channel (with noise intensity = 1) with two equally likely messages transmitted via waveforms $w_0(t) = -w_1(t) = \sqrt{E}1\{|t| < \frac{1}{2}\}$.

(a) (3 pts) At the receiver suppose we pass the received signal $R(t)$ through a filter with impulse response $h(t) = 1\{|t| < \frac{1}{2}\}$, sample the filter output at $t_0 = 0$, and, with $Y$ denoting the value of the sample, decide $\hat{H} = 1$ if $Y < 0$, $\hat{H} = 0$ if $Y \geq 0$. Is this receiver optimal? What is the probability of error?

We are asked to design a receiver who does not get to observe $R(t)$ but observes the output $S(t)$ of a filter whose input is $R(t)$ and whose impulse response is $1\{|t| < 1\}$. We can sample $S(t)$ at any number of time instants $t_1, \ldots, t_n$, and base our decision on the values $Y_1 = S(t_1), \ldots, Y_n = S(t_n)$.

(b) (2 pts) Suppose we choose $n = 1$, and $t_1 = 0$. What is the distribution of $Y_1$ given $H = i, i = 0, 1$?

(c) (3 pts) What is the optimal choice of $\hat{H}(Y_1)$ and what is the corresponding probability of error?

(d) (3 pts) Suppose we choose $n = 2$, with $t_1 = -1/2, t_2 = 1/2$. What is the optimal choice of $\hat{H}(Y_1, Y_2)$ and what is the probability of error?

Hint: $Y_1 + Y_2$ is a sufficient statistic.
Problem 4. (10 points)
Consider a communication system for the AWGN channel with noise intensity $N_0/2$ with four equally likely messages, and suppose the waveforms $w_1, \ldots, w_4$ have all unit norm and that $\langle w_i, w_k \rangle = \alpha$ for all $i \neq k$.

(a) (2 pts) Express $\|w_1 + w_2 + w_3 + w_4\|^2$ in terms of $\alpha$, and show that $-1/3 \leq \alpha \leq 1$.

(b) (3 pts) Let $\{\tilde{w}_1, \ldots, \tilde{w}_4\}$ be obtained by a translation of $\{w_1, \ldots, w_4\}$ so that the new signal set is of minimal average energy. Do $\tilde{w}_1, \ldots, \tilde{w}_4$ all have the same energy? If so, what is this energy in terms of $\alpha$?

(c) (2 pts) Is there a common value of $\langle \tilde{w}_i, \tilde{w}_k \rangle$ for $i \neq k$? If so, what is the common value in terms of $\alpha$?

(d) (3 pts) Let $c_1, c_2, c_3, c_4$ be the the corners of a regular tetrahedron in $\mathbb{R}^3$, centered at the origin. I.e., (i) $\|c_i\|^2 = A^2$ for all $i$, (ii) $\langle c_i, c_k \rangle = -A^2/3$ for all $i \neq k$. (As a consequence, $c_1 + c_2 + c_3 + c_4 = 0$.)

Let $e_{\text{tetra}}(A)$ denote the error probability of the MAP decoder that observes $Y = c_i + Z$ where $Z$ is $\mathcal{N}(0, I_3)$ where each of the four $c_i$'s are equally likely. Express the probability of error of the communication system (the system which uses the waveforms $w_1, w_2, w_3, w_4$) described at the start of the problem in terms of $\alpha$, $N_0$ and $e_{\text{tetra}}(\cdot)$.

*Hint:* No lengthy computations are needed.