Handout 17 | Principles of Digital Communications
Solutions to Midterm exam | Apr. 22, 2023

PROBLEM 1. (10 points)
Consider a hypothesis testing problem where the hypothesis H can take the values 0 or 1 with equal probability. The observation $Y = (Y_1, Y_2)$, when $H = i$, is given by

$$Y = c_i + Z,$$

where $c_0 = (1, 2)$ and $c_1 = (2, 1)$ are vectors in $\mathbb{R}^2$ and Z is Gaussian, zero mean, with covariance matrix $K = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

Let α be a real number and let $T = (1-\alpha)Y_1 + \alpha Y_2$ be a 1-dimensional statistic.

(a) (2 pts) What is the probability distribution of the statistic T when $H = i$, $i = 0, 1$?

Solution: Defining $v = (1-\alpha, \alpha)$ and $Z = (Z_1, Z_2)$, when $H = i$, $T = u_i + W$, where $u_i = \langle v, c_i \rangle$, and $W = \langle v, Z \rangle = (1-\alpha)Z_1 + \alpha Z_2$. For $i = 0$, we have $u_0 = \langle v, c_0 \rangle = (1-\alpha) + 2\alpha = 1 + \alpha$, and for $i = 1$, we have $u_1 = \langle v, c_1 \rangle = 2(1-\alpha) + \alpha = 2 - \alpha$. For either value of $H = i$, T is a Gaussian random variable with mean u_i and variance the same as W, given by $(1-\alpha)^2 + 2\alpha^2 = \sigma^2$, say. Hence, when $H = 0$, $T \sim \mathcal{N}(1+\alpha, \sigma^2)$ and when $H = 1$, $T \sim \mathcal{N}(2-\alpha, \sigma^2)$.

(b) (2 pts) Consider the MAP decision rule *based only on the statistic* T. What is the error probability for this rule?

Solution: With the same notation as in part (a), note that W (which is the noise associated with the statistic T) is a zero mean Gaussian with variance $\sigma^2 = (1-\alpha)^2 + 2\alpha^2$. The error probability is thus $Q\left(\frac{d}{2\sigma}\right)$ where $d = |u_0 - u_1| = |1 - 2\alpha|$.

(c) (3 pts) Which choice of α will minimize the error probability in (b)?

Solution: Minimizing the error probability in (b), is equivalent to maximizing $d^2/\sigma^2 = (1-2\alpha)^2/(1-2\alpha+3\alpha^2)$ (since $|x| \mapsto x^2$ is an increasing mapping and $Q(\cdot)$ is a decreasing function). On differentiating, we find that $\alpha = -1$ is the maximizer.

(d) (3 pts) Is the statistic T, with the α of (c), a sufficient statistic?

Solution: Yes. The likelihood ratio for the observation $y = (y_1, y_2)$ is equal to

$$\begin{aligned}
\frac{f_{Y|H}(y|0)}{f_{Y|H}(y|1)} = \frac{f_Z(y-c_0)}{f_Z(y-c_1)} &= \frac{f_{Z_1}(y_1-1)f_{Z_2}(y_2-2)}{f_{Z_1}(y_1-2)f_{Z_2}(y_2-1)} \\
&= \frac{\exp\left(-\frac{(y_1-1)^2}{2}\right)\exp\left(-\frac{(y_2-2)^2}{4}\right)}{\exp\left(-\frac{(y_1-2)^2}{2}\right)\exp\left(-\frac{(y_2-1)^2}{4}\right)} \\
&= \exp\left(y_1 - \frac{1}{2} + y_2 - 1 - 2y_1 + 2 - \frac{y_2}{2} + \frac{1}{4}\right) \\
&= \exp\left(\frac{3}{4} - \frac{1}{2}(2y_1 - y_2)\right),
\end{aligned}$$

which is a function of y only through $2y_1 - y_2$, which is exactly the statistic T with $\alpha = -1$.

Remark: In general, when $Y = c_i + Z$ with Z Gaussian and covariance K, $T = \langle v, Y \rangle$ is a statistic, and v is chosen to maximize the SNR $= \frac{\langle v, c_0 - c_1 \rangle^2}{\langle v, Kv \rangle}$, then T is a sufficient statistic. This problem is only a special case.

PROBLEM 2. (10 points)

Consider a communication channel with input (x_1, x_2) in $\mathbb{R}^2$, and output (Y_1, Y_2) in $\mathbb{R}^2$ given by

$$Y_1 = A_1 x_1 - A_2 x_2 + Z_1$$
$$Y_2 = A_2 x_1 + A_1 x_2 + Z_2$$

where A_1, A_2, Z_1, Z_2 are all i.i.d. $\mathcal{N}(0, 1)$ random variables.

(a) (2 pts) Observe that (Y_1, Y_2) is a Gaussian vector for any given (x_1, x_2). Find its mean and covariance matrix in terms of x_1, x_2.

Solution: Since A_1, A_2, Z_1, Z_2 are all zero mean random variables, (Y_1, Y_2) is also zero mean. Since A_1, A_2, Z_1, Z_2 are independent, the variances of Y_1 and Y_2 are given by

$$\begin{aligned}\mathrm{Var}(Y_1) &= x_1^2 \mathrm{Var}(A_1) + x_2^2 \mathrm{Var}(A_2) + \mathrm{Var}(Z_1) = 1 + x_1^2 + x_2^2, \\ \mathrm{Var}(Y_2) &= x_1^2 \mathrm{Var}(A_2) + x_2^2 \mathrm{Var}(A_1) + \mathrm{Var}(Z_2) = 1 + x_1^2 + x_2^2.\end{aligned}$$

Further, the covariance between Y_1 and Y_2 is equal to zero, as

$$\mathbb{E}[Y_1 Y_2] = x_1 x_2 (\mathbb{E}[A_1^2] - \mathbb{E}[A_2^2]) + (x_1^2 - x_2^2)\mathbb{E}[A_1]\mathbb{E}[A_2] + \mathbb{E}[Z_1](\dots) + +\mathbb{E}[Z_2](\dots) = 0.$$

Hence $(Y_1, Y_2) \sim \mathcal{N}\left((0, 0),\ (1 + x_1^2 + x_2^2) I_2\right) = \mathcal{N}\left((0, 0),\ (1 + \|x\|^2) I_2\right)$.

(b) (3 pts) Suppose $c_1, \dots, c_m$ are m vectors in $\mathbb{R}^2$, and that when the message H equals i, the vector c_i is input to the communication channel above. The receiver, from the observation (Y_1, Y_2) tries to guess the value of H. Show that, no matter how the vectors $c_1, \dots, c_m$ are chosen, $T = Y_1^2 + Y_2^2$ is a sufficient statistic.

Solution: The pdf of the observation $y = (y_1, y_2)$ when $H = i$ is given by

$$f_{Y|H}(y|i) = \frac{1}{2\pi(1 + \|c_i\|^2)} \exp\left(-\frac{y_1^2 + y_2^2}{2(1 + \|c_i\|^2)}\right),$$

which depends only on y only through $y_1^2 + y_2^2$, hence $T = Y_1^2 + Y_2^2$ is a sufficient statistic.

(c) (2 pts) Suppose $m = 4$, and $c_1 = (5, 0), c_2 = (0, 5), c_3 = (3, 4), c_4 = (4, 3)$. All four messages are equally likely. What is the probability of error of the MAP decoder?

Solution: Since all the c_i's have the same norm, the output is independent of the input, thus the probability of error $= 3/4$ (equivalent to a random guess between the four options).

(d) (3 pts) Consider four designs, all with $m = 2$, all with equally likely messages:

1. $c_1 = (0, 0), c_2 = (10, 0)$
2. $c_1 = (8, 6), c_2 = (0, 0)$
3. $c_1 = (0, 0), c_2 = (5, 0)$
4. $c_1 = (8, 6), c_2 = (10, 0)$

With P_1, P_2, P_3, P_4 denoting the error probability with MAP decoding of these systems, how will $P_1, \dots, P_4$ be ordered? What is the value of P_4?

Solution: $P_1 = P_2 < P_3 < P_4 = 1/2$.

$P_1 = P_2$ because in both cases, one message has norm 0 and the other has norm 10. $P_1 < P_3$ because the distance between the message constellation is more in design 1. $P_4 = 1/2$ for the same reason as part (c), and $P_3 < P_4$ because we can definitely do better than a random guess between the two messages in design 3.

Remark: Note that the channel is the "real" equivalent of the "complex" channel $Y = Ax + Z$. Since A and Z are both circularly symmetric, is it not a surprise that $|Y|^2$ is a sufficient statistic and the input influences the output only via $|x|^2$.

Problem 3. (11 points)

Consider a communication system over an additive white Gaussian noise channel (with noise intensity = 1) with two equally likely messages transmitted via waveforms $w_0(t) = -w_1(t) = \sqrt{\mathcal{E}}\mathbb{1}\{|t| < \frac{1}{2}\}$.

(a) (3 pts) At the receiver suppose we pass the received signal $R(t)$ through a filter with impulse response $h(t) = \mathbb{1}\{|t| < \frac{1}{2}\}$, sample the filter output at $t_0 = 0$, and, with Y denoting the value of the sample, decide $\hat{H} = 1$ if $Y < 0$, $\hat{H} = 0$ if $Y \geq 0$. Is this receiver optimal? What is the probability of error?

Solution: Yes, this receiver does exactly what the optimal MAP receiver would do (compute the inner product of R with the orthonormal basis functions). To see this, observe that the orthonormal basis function is $\psi(t) = \mathbb{1}\{|t| < \frac{1}{2}\}$, which is exactly equal to $h(T - t)$ with $T = 0$. Hence, sampling the output of the filter at $t = T = 0$ gives us the inner product of R with ψ.

We thus have $Y = c_i + Z$, where $Y = \langle R, \psi \rangle$, $c_i = \langle w_i, \psi \rangle$ and $Z = \langle N, \psi \rangle$, where $N(t)$ is AWGN with noise intensity 1. It is easy to see that $c_i = \sqrt{\mathcal{E}}$ for $i = 0$ and $-\sqrt{\mathcal{E}}$ for $i = 1$, and Z is a Gaussian random variable with mean 0 and variance 1. Hence the probability of error is $Q(\sqrt{\mathcal{E}})$.

We are asked to design a receiver who does not get to observe $R(t)$ but observes the output $S(t)$ of a filter whose input is $R(t)$ and whose impulse response is $\mathbb{1}\{|t| < 1\}$. We can sample $S(t)$ at any number of time instants $t_1, \dots, t_n$, and base our decision on the values $Y_1 = S(t_1), \dots, Y_n = S(t_n)$.

(b) (2 pts) Suppose we choose $n = 1$, and $t_1 = 0$. What is the distribution of Y_1 given $H = i$, $i = 0, 1$?

Solution: Let $h'(t) = \mathbb{1}\{|t| < 1\}$ be the new impulse response. Then $Y_1 = c'_i + Z'$, where

$$c'_i = \int_{\mathbb{R}} w_i(t)h'(-t)\,\mathrm{d}t = \begin{cases} \int_{-\frac{1}{2}}^{\frac{1}{2}} \sqrt{\mathcal{E}}\,\mathrm{d}t & \text{if } i = 0 \\ \int_{-\frac{1}{2}}^{\frac{1}{2}} -\sqrt{\mathcal{E}}\,\mathrm{d}t & \text{if } i = 1 \end{cases} = \begin{cases} \sqrt{\mathcal{E}} & \text{if } i = 0 \\ -\sqrt{\mathcal{E}} & \text{if } i = 1 \end{cases}$$

$$Z' = \int_{\mathbb{R}} N(t)h'(-t)\,\mathrm{d}t = \int_{-1}^{1} N(t)\,\mathrm{d}t \sim \mathcal{N}(0, 2),$$

where the last step follows since $\|h'\|^2 = 2$. Hence, $Y_1 \sim \mathcal{N}(\sqrt{\mathcal{E}}, 2)$ for $i = 0$ and $Y_1 \sim \mathcal{N}(-\sqrt{\mathcal{E}}, 2)$ for $i = 1$.

(c) (3 pts) What is the optimal choice of $\hat{H}(Y_1)$ and what is the corresponding probability of error?

Solution: Just as in part (a), the optimal choice of $\hat{H}(Y_1)$ is to decide $\hat{H}(Y_1) = 1$ if $Y_1 < 0$, $\hat{H}(Y_1) = 0$ if $Y_1 \geq 0$, and since the noise variance is now 2, the error probability is $Q\left(\sqrt{\frac{\mathcal{E}}{2}}\right)$.

(d) (3 pts) Suppose we choose $n = 2$, with $t_1 = -1/2, t_2 = 1/2$. What is the optimal choice of $\hat{H}(Y_1, Y_2)$ and what is the probability of error?
Hint: $Y_1 + Y_2$ is a sufficient statistic.

Solution: Observe that $Y_1 + Y_2 = \int_{\mathbb{R}} R(t) \left(h' \left(-\frac{1}{2} - t \right) + h' \left(\frac{1}{2} - t \right) \right) \mathrm{d}t$, then define

$$h''(t) = h'\left(-\tfrac{1}{2} - t\right) + h'\left(\tfrac{1}{2} - t\right) = \begin{cases} 1 & \text{if } -\frac{3}{2} < t < -\frac{1}{2} \\ 2 & \text{if } -\frac{1}{2} < t < \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} < t < \frac{3}{2} \\ 0 & \text{else} \end{cases}.$$

Hence we have $Y_1 + Y_2 = c_i'' + Z''$, where

$$\begin{aligned} c_i'' &= \int_{\mathbb{R}} w_i(t) h''(t)\, \mathrm{d}t \\ &= \begin{cases} \int_{-\frac{1}{2}}^{\frac{1}{2}} 2\sqrt{\mathcal{E}}\, \mathrm{d}t & \text{if } i = 0 \\ \int_{-\frac{1}{2}}^{\frac{1}{2}} -2\sqrt{\mathcal{E}}\, \mathrm{d}t & \text{if } i = 1 \end{cases} = \begin{cases} 2\sqrt{\mathcal{E}} & \text{if } i = 0 \\ -2\sqrt{\mathcal{E}} & \text{if } i = 1 \end{cases}, \text{ and} \\ Z' &= \int_{\mathbb{R}} N(t) h''(t)\, \mathrm{d}t \sim \mathcal{N}(0, 6), \end{aligned}$$

where the last step follows since $\|h''\|^2 = 6$. Thus we have that the optimal choice of $\hat{H}(Y_1, Y_2)$ is to decide $\hat{H}(Y_1, Y_2) = 1$ if $Y_1 + Y_2 < 0$, $\hat{H}(Y_1) = 0$ if $Y_1 + Y_2 \geq 0$, and the error probability is $Q\left(\frac{2\sqrt{\mathcal{E}}}{\sqrt{6}}\right) = Q\left(\sqrt{\frac{2\mathcal{E}}{3}}\right)$.

Remark: Let $\hat{X}(f)$ denote the Fourier transform of any signal $X(t)$. The signal that we would ideally like to have as in part (a) is the inverse Fourier transform of $\hat{R}(f)\hat{h}(f)$, with $\hat{h}(f) = \mathrm{sinc}(f)$, and then sample the output at $t = 0$ to obtain the sufficient statistic Y. However, we are only able to observe the signal $S(t)$ with Fourier transform $\hat{S}(f) = \hat{R}(f)\hat{h}'(f)$, where $\hat{h}' = 2\mathrm{sinc}(2f)$. Therefore, in principle, by passing $S(t)$ through another filter with frequency response (i.e., the Fourier transform of the impulse response) $\hat{g}(f) = \frac{\hat{h}(f)}{\hat{h}'(f)} = \frac{1}{2\cos(\pi f)}$, we should be able to obtain the desired signal. Since $\hat{g}(f)$ is periodic with period 2, the impulse response $g(t)$ is of the form $\sum_{n \in \mathbb{Z}} c_n \delta\left(t - \frac{n}{2}\right)$ for some c_n. This is equivalent to sampling $S(t)$ at the instances $t = \frac{n}{2}$, scaling them by c_n, and summing the resulting quantities. The parts (c) and (d) look at the error performance when we truncate the sum to $n = 1$ and $n = 2$ terms respectively. By continuing to an infinite number of terms, we can, in principle, we recover the same error probability as part (a).

Problem 4. (10 points)
Consider a communication system for the AWGN channel with noise intensity $N_0/2$ with four equally likely messages, and suppose the waveforms $w_1, \dots, w_4$ have all unit norm and that $\langle w_i, w_k \rangle = \alpha$ for all $i \neq k$.

(a) (2 pts) Express $\|w_1 + w_2 + w_3 + w_4\|^2$ in terms of α, and show that $-1/3 \le \alpha \le 1$.

Solution: Expanding $\|w_1 + w_2 + w_3 + w_4\|^2$, we have

$$\|w_1 + w_2 + w_3 + w_4\|^2 = \sum_{1 \le i \le 4} \|w_i\|^2 + \sum_{1 \le i \neq j \le 4} \langle w_i, w_j \rangle = 4 + 12\alpha,$$

and since $4 + 12\alpha = \|w_1 + w_2 + w_3 + w_4\|^2 \ge 0$, we have $\alpha \ge -1/3$. Furthermore, by the Cauchy-Schwarz inequality, we have, for $i \neq k$,

$$\alpha = \langle w_i, w_k \rangle \le \|w_i\| \|w_k\| = 1.$$

(b) (3 pts) Let $\{\tilde{w}_1, \dots, \tilde{w}_4\}$ be obtained by a translation of $\{w_1, \dots, w_4\}$ so that the new signal set is of minimal average energy. Do $\tilde{w}_1, \dots, \tilde{w}_4$ all have the same energy? If so, what is this energy in terms of α?

Solution: The minimal average energy set of waveforms $\{\tilde{w}_1, \tilde{w}_2, \tilde{w}_3, \tilde{w}_4\}$ is formed by subtracting the arithmetic mean $m = \frac{1}{4}(w_1 + w_2 + w_3 + w_4)$ from each waveform, i.e., $\tilde{w}_i = w_i - m$. Then,

$$\begin{aligned}
\|\tilde{w}_i\|^2 &= \|w_i - m\|^2 = \|w_i\|^2 + \|m\|^2 - 2\langle w_i, m \rangle \\
&= 1 + \left\| \frac{1}{4}(w_1 + w_2 + w_3 + w_4) \right\|^2 - 2 \left\langle w_i, \frac{1}{4} \sum_{1 \le j \le 4} w_j \right\rangle \\
&= 1 + \frac{1}{16}(4 + 12\alpha) - \frac{2}{4}(1 + 3\alpha) = \frac{3}{4}(1 - \alpha)
\end{aligned}$$

for all i. Hence all the minimal energy waveforms $\tilde{w}_1, \tilde{w}_2, \tilde{w}_3, \tilde{w}_4$ have the same energy, given by $\frac{3}{4}(1 - \alpha)$.

(c) (2 pts) Is there a common value of $\langle \tilde{w}_i, \tilde{w}_k \rangle$ for $i \neq k$? If so, what is the common value in terms of α?

Solution: For any $i \neq k$, we have

$$\begin{aligned}
\langle \tilde{w}_i, \tilde{w}_k \rangle &= \langle w_i - m, w_k - m \rangle = \langle w_i, w_k \rangle + \|m\|^2 - \langle w_i, m \rangle - \langle w_k, m \rangle \\
&= \alpha + \left\| \frac{1}{4}(w_1 + w_2 + w_3 + w_4) \right\|^2 - \left\langle w_i, \frac{1}{4} \sum_{1 \le j \le 4} w_j \right\rangle - \left\langle w_k, \frac{1}{4} \sum_{1 \le j \le 4} w_j \right\rangle \\
&= \alpha + \frac{1}{16}(4 + 12\alpha) - \frac{2}{4}(1 + 3\alpha) = \frac{1}{4}(\alpha - 1).
\end{aligned}$$

(d) (3 pts) Let c_1, c_2, c_3, c_4 be the the corners of a regular tetrahedron in $\mathbb{R}^3$, centered at the origin. I.e., (i) $\|c_i\|^2 = A^2$ for all i, (ii) $\langle c_i, c_k \rangle = -A^2/3$ for all $i \neq k$. (As a consequence, $c_1 + c_2 + c_3 + c_4 = 0$.)

Let $e_{\text{tetra}}(A)$ denote the error probability of the MAP decoder that observes $Y = c_i + Z$ where Z is $\mathcal{N}(0, I_3)$ where each of the four c_i's are equally likely. Express the probability of error of the communication system (the system which uses the waveforms w_1, w_2, w_3, w_4) described at the start of the problem in terms of α, N_0 and $e_{\text{tetra}}(\cdot)$.

Hint: No lengthy computations are needed.

Solution: First observe that the waveform set $\tilde{\mathcal{W}} = \{\tilde{w}_1, \dots, \tilde{w}_4\}$ is an isometric transformation of $\mathcal{W} = \{w_1, \dots, w_4\}$, hence the error probability of the system using $\mathcal{W}$ is identical to that using $\tilde{\mathcal{W}}$. Observe that $\tilde{\mathcal{W}}$ has dimension 3, since $\tilde{w}_1 + \cdots + \tilde{w}_4 = 0$, and let $\Psi = \{\psi_1, \psi_2, \psi_3\}$ be an orthonormal basis for the waveform set.

Given the received signal $R(t) = \tilde{w}_i(t) + N(t)$, where $N(t)$ is AWGN with noise intensity $\frac{N_0}{2}$, we compute the sufficient statistic $Y = (\langle R, \psi_1\rangle, \langle R, \psi_2\rangle, \langle R, \psi_3\rangle) = c_i + Z$, where $c_i = (\langle \tilde{w}_i, \psi_1\rangle, \langle \tilde{w}_i, \psi_2\rangle, \langle \tilde{w}_i, \psi_3\rangle)$ and $Z = (\langle N, \psi_1\rangle, \langle N, \psi_2\rangle, \langle N, \psi_3\rangle) \sim \mathcal{N}\left(0, \frac{N_0}{2} I_3\right)$.

Define $\tilde{Y} = \frac{Y}{\sqrt{N_0/2}} = \tilde{c}_i + \tilde{Z}$ with $\tilde{c}_i = \frac{c_i}{\sqrt{N_0/2}}$ and $\tilde{Z} = \frac{Z}{\sqrt{N_0/2}} \sim \mathcal{N}(0, I_3)$. Setting $E = \frac{3}{4}(1-\alpha)$, we have $\|\tilde{c}_i\|^2 = \frac{1}{N_0/2}\|\tilde{w}_i\|^2 = \frac{2E}{N_0}$, $\langle \tilde{c}_i, \tilde{c}_k\rangle = \frac{1}{N_0/2}\langle \tilde{w}_i, \tilde{w}_k\rangle = -\frac{2E}{N_0}\Big/3$ for all $i \neq k$, i.e., $\tilde{c}_i$ are corners of a regular tetrahedron in $\mathbb{R}^3$ centered at the origin as described in the problem, with $A = \sqrt{\frac{2E}{N_0}} = \frac{3(1-\alpha)}{2N_0}$. Hence the error probability of the system is $e_{\text{tetra}}\left(\sqrt{\frac{3(1-\alpha)}{2N_0}}\right)$.

Remark: This problem is related to the *simplex conjecture*, which states that the optimal choice of M signal vectors in AWGN, with an average energy constraint but no constraint on the dimension of the signal set, is the vertices of the $(M-1)$-dimensional regular simplex (e.g., regular tetradehedron for $M = 4$, equilateral triangle for $M = 3$). A counter example has been shown for $M \geq 7$, and hence this conjecture is not true in general. Refer to M. Steiner, "The strong simplex conjecture is false," in *IEEE Transactions on Information Theory*, vol. 40, no. 3, pp. 721-731, May 1994 (available online at https://ieeexplore.ieee.org/abstract/document/335884), for more details.