SOLUTION 1.

(a)

\[ R_\xi(\tau) = \int_{-\infty}^{\infty} \xi(t+\tau)\xi^*(t) \, dt = \langle \xi(t+\tau), \xi(t) \rangle \]

\[ \leq (1) \| \xi(t+\tau) \| \cdot \| \xi(t) \| = \| \xi \| \cdot \| \xi \| = \| \xi \|^2 = R_\xi(0), \]

where (1) follows from the Cauchy–Schwarz inequality and (2) from the fact that \( R_\xi(0) = \int_{-\infty}^{\infty} \xi(t)\xi^*(t) \, dt = \| \xi \|^2. \)

(b)

\[ R_\xi(-\tau) = \int_{-\infty}^{\infty} \xi(t-\tau)\xi^*(t) \, dt = \left( \int_{-\infty}^{\infty} \xi(t)\xi^*(t-\tau) \, dt \right)^* \]

\[ t \rightarrow t+\tau \rightarrow R^*_\xi(\tau). \]

(c)

\[ R_\xi(\tau) = \int_{-\infty}^{\infty} \xi(t+\tau)\xi^*(t) \, dt \]

\[ t \rightarrow t-\tau \rightarrow \int_{-\infty}^{\infty} \xi(t)\xi^*(t-\tau) \, dt \]

\[ = \xi(\tau) \ast \xi^*(-\tau). \]

(d) By Parseval’s identity, we have

\[ R_\xi(\tau) = \langle \xi(t+\tau), \xi(t) \rangle \]

\[ = \langle \xi_\mathcal{F}(f)e^{j2\pi f \tau}, \xi_\mathcal{F}(f) \rangle \]

\[ = \int_{-\infty}^{\infty} \xi_\mathcal{F}(f)\xi^*_\mathcal{F}(f)e^{j2\pi f \tau} \, df \]

\[ = \int_{-\infty}^{\infty} \| \xi_\mathcal{F}(f) \|^2 e^{j2\pi f \tau} \, df, \]

which is the inverse Fourier transform of \( |\xi_\mathcal{F}(f)|^2. \)
Solution 2.

(a) We have
\[ y(t) = \int_{-\infty}^{\infty} w(\tau) \psi(\tau - t) d\tau. \]

The samples of this waveform at multiples of \( T \) are
\[
y(mT) = \int_{-\infty}^{\infty} w(\tau) \psi(\tau - mT) d\tau
= \int_{-\infty}^{\infty} \left( \sum_{k=1}^{K} d_k \psi(\tau - kT) \right) \psi(\tau - mT) d\tau
= \sum_{k=1}^{K} d_k \int_{-\infty}^{\infty} \psi(\tau - kT) \psi(\tau - mT) d\tau
= \sum_{k=1}^{K} d_k \mathbb{1} \{ k = m \}
= d_m.
\]

(b) Let \( \tilde{w}(t) \) be the channel output. Then, \( \tilde{y}(t) \) is \( \tilde{w}(t) \) filtered by \( \psi(-t) \). We have
\[
\tilde{w}(t) = w(t) + \rho w(t - T)
\]
and
\[
\tilde{y}(t) = \int_{-\infty}^{\infty} \tilde{w}(\tau) \psi(\tau - t) d\tau.
\]
The samples of this waveform at multiples of \( T \) are
\[
\tilde{y}(mT) = \int_{-\infty}^{\infty} \tilde{w}(\tau) \psi(\tau - mT) d\tau
= \int_{-\infty}^{\infty} \left[ w(\tau) + \rho w(\tau - T) \right] \psi(\tau - mT) d\tau
= \int_{-\infty}^{\infty} \left( \sum_{k=1}^{K} d_k \psi(\tau - kT) \right) \psi(\tau - mT) d\tau + \\
\rho \int_{-\infty}^{\infty} \left( \sum_{k=1}^{K} d_k \psi(\tau - T - kT) \right) \psi(\tau - mT) d\tau
= \sum_{k=1}^{K} d_k \mathbb{1} \{ k = m \} + \rho \sum_{k=1}^{K} d_k \mathbb{1} \{ k = m - 1 \}
= d_m + \rho d_{m-1}.
\]

(c) From the symmetry of the problem, we have
\[ P_e = P_e(1) = P_e(-1). \]
\[
P_2(1) = \Pr\{\hat{D}_k = -1|D_k = 1, D_{k-1} = -1\} \Pr\{D_{k-1} = -1\} + \\
\Pr\{D_k = -1|D_k = 1, D_{k-1} = 1\} \Pr\{D_{k-1} = 1\}
\]

\[
= \frac{1}{2} \left( \Pr\{Y_k < 0|D_k = 1, D_{k-1} = -1\} + \Pr\{Y_k < 0|D_k = 1, D_{k-1} = 1\} \right)
\]

\[
= \frac{1}{2} \left( \Pr\{1 - \alpha + Z_k < 0\} + \Pr\{1 + \alpha + Z_k < 0\} \right)
\]

\[
= \frac{1}{2} \left( \Pr\{Z_k < -1 + \alpha\} + \Pr\{Z_k < -1 - \alpha\} \right)
\]

\[
= \frac{1}{2} \left[ Q \left( \frac{1 - \alpha}{\sigma} \right) + Q \left( \frac{1 + \alpha}{\sigma} \right) \right].
\]

**Solution 3.**

(a) We can easily see that

\[
\mathbb{E}[X_i|X_{i-1}] = \frac{1}{2} X_{i-1} + \frac{1}{2} (-X_{i-1}) = 0.
\]

Consequently (using the law of total expectation)

\[
\mathbb{E}[X_i] = \mathbb{E}[\mathbb{E}[X_i|X_{i-1}]] = 0.
\]

Therefore,

\[
K_X[k] = \mathbb{E}[(X_i - \mathbb{E}[X_i])(X_{i-k} - \mathbb{E}[X_{i-k}])^*] = \mathbb{E}[X_i X_{i-k}^*]
\]

Moreover, using the fact that \(X_i = X_{i-1} \times (-1)^{D_i}\) repeatedly, we can write

\[
X_i = X_{i-k} \times \prod_{j=i-k+1}^{i} (-1)^{D_j}
\]

Thus,

\[
K_X[k] = \mathbb{E}[X_i X_{i-k}^*]
\]

\[
= \mathbb{E} \left[ X_{i-k} \prod_{j=i-k+1}^{i} (-1)^{D_j} X_{i-k}^* \right]
\]

\[
= \mathbb{E} \left[ X_{i-k} \prod_{j=i-k+1}^{i} (-1)^{D_j} \right] \quad \mathbb{E}[X_i X_{i-k}^*]
\]

\[
= \mathbb{E} \prod_{j=i-k+1}^{i} \mathbb{E}[(-1)^{D_j}]
\]

\[
= \begin{cases} 
\mathbb{E} & \text{if } k = 0, \\
0 & \text{otherwise}.
\end{cases}
\]

where (a) follows from the independence of data bits \(\{D_i\}\) and (b) since \(\mathbb{E}[(-1)^{D_i}] = 0\).

(b) By sampling the signal at the output of the matched filter, \(Y(t)\), at multiples of \(T\), we obtain

\[
Y(iT) = X_i + Z_i,
\]
where $Z_i$ is normally distributed with zero mean and variance $N_0/2$. By looking at the definition of $X_i$, we see that it is equal to $X_{i-1}$ if $D_i = 0$ and equal to $-X_{i-1}$ if $D_i = 1$. Therefore a simple decoder estimates that $\hat{D}_i = 0$ if $Y_i$ and $Y_{i-1}$ have the same sign, and $\hat{D}_i = 1$ otherwise. This is equivalent to

$$ Y_i Y_{i-1} \begin{cases} \geq 0, & D_i = 0 \\ < 0, & D_i = 1 \end{cases} $$

(c) We first compute the error probability when $D_i = 0$. If $X_i - 1 = \sqrt{E}$, then $X_i = \sqrt{E}$. When we decode, we will make an error if the signal $(Y_{i-1}, Y_i)^T$ is in the second or fourth quadrants (shaded regions in the following figure).

Due to the symmetry of the problem, the probability for this to happen is two times the probability for $(Y_{i-1}, Y_i)^T$ to be in the second quadrant:

$$ \Pr\{Z_{i-1} < -\sqrt{E} \cap Z_i > -\sqrt{E} \} = Q\left(\sqrt{\frac{E}{N_0/2}}\right) Q\left(-\sqrt{\frac{E}{N_0/2}}\right), $$

so,

$$ P_e(D_i = 0|D_{i-1} = 0) = 2Q\left(\sqrt{\frac{E}{N_0/2}}\right) Q\left(-\sqrt{\frac{E}{N_0/2}}\right). $$

Again, due to the symmetry of the problem,

$$ P_e(D_i = 0|D_{i-1} = 1) = P_e(D_i = 0|D_{i-1} = 0) = P_e(D_i = 0), $$

and

$$ P_e(D_i = 1) = P_e(D_i = 0); $$

hence

$$ P_e = 2Q\left(\sqrt{\frac{E}{N_0/2}}\right) Q\left(-\sqrt{\frac{E}{N_0/2}}\right). $$

**Solution 4.** Because $\psi(t)$ is real, its Fourier transform is conjugate symmetric ($\psi_F(f) = \psi^*_F(-f)$).

From the condition $\int \psi(t - kT)\psi(t - lT)dt = 1\{k = l\}$ for every pair $k, l$, it follows that $|\psi_F(f)|^2$ satisfies Nyquist’s criterion with parameter $T$, $\sum_{k \in \mathbb{Z}} |\psi_F(f - k/T)|^2 = T$. On the other hand, since $\psi_F(f) = 0$ for $|f| > \frac{1}{T}$, $|\psi_F(f)|^2$ must have band-edge symmetry.

Putting everything together, we obtain the complete plot of $|\psi_F(f)|^2$. 

\[\text{4}\]
Solution 5. From Theorem 5.6, we know that \( \{\psi(t - jT)\}_{j=-\infty}^{\infty} \) is an orthonormal set if and only if
\[
\sum_{k \in \mathbb{Z}} |\psi_f(f - \frac{k}{T})|^2 = T.
\]

(a) \[
\sum_{k \in \mathbb{Z}} T \mathbb{1}[\frac{k}{T} - \frac{1}{T} \leq f \leq \frac{k}{T} + \frac{1}{T}] (f) = T \Rightarrow \text{The Nyquist criterion is satisfied}
\]
\[
\Rightarrow \psi(t) \text{ is orthonormal to its time-translates by multiples of } T.
\]

(b) \[
\sum_{k \in \mathbb{Z}} \frac{T}{2} \mathbb{1}[\frac{k}{T} - \frac{1}{T} \leq f \leq \frac{k}{T} + \frac{1}{T}] (f) = T \Rightarrow \text{The Nyquist criterion is satisfied}
\]
\[
\Rightarrow \psi(t) \text{ is orthonormal to its time-translates by multiples of } T.
\]

(c) Because \( |\psi_f(f)|^2 \) vanishes outside \( \left[ -\frac{1}{T}, \frac{1}{T} \right] \), we verify whether the band-edge symmetry is fulfilled, which is the case. Hence, the Nyquist criterion is satisfied and \( \psi(t) \) is orthonormal to its time-translates by multiples of \( T \). Note: the same reasoning can be applied to (b).

(d) \( \psi_f(f) \) is a sinc function, therefore \( \psi(t) \) is a box function, equal to \( \frac{1}{T} \mathbb{1}[\frac{-T}{2}, \frac{T}{2}](t) \). This is orthogonal to its time-translates by multiples of \( T \), but does not have unit norm (unless \( T = 1 \)): \( \int_{-\infty}^{\infty} |\psi(t)|^2 \, dt = \frac{1}{T} \).

Solution 6.

(a) We pass \( R(t) \) through a whitening filter \( h(t) \) such that the output \( R'(t) \) looks like the output of an AWGN channel. After this step we are facing a familiar situation and can implement a matched filter receiver. The receiver architecture is shown below:

![Receiver Architecture Diagram]
Let \( N'(t) = \int N(\alpha)h(t - \alpha) \, d\alpha \) be the noise at the output of the whitening filter. We want to select the filter \( h(t) \) such that \( \frac{N_0}{2} = G(f)|h_f(f)|^2 \), i.e.,

\[
|h_f(f)|^2 = \frac{N_0}{2G(f)}.
\]

The output of the filter is

\[
R'(t) = \int R(\alpha)h(t - \alpha) \, d\alpha = \int w_i(\alpha)h(t - \alpha) \, d\alpha + \int N(\alpha)h(t - \alpha) \, d\alpha = w'_i(t) + N'(t),
\]

where \( N'(t) \) is white Gaussian noise and \( w'_i(t) = \int w_i(\alpha)h(t - \alpha) \, d\alpha \). We need to design the matched filter for the signals \( w'_i(t) \).

(b) To minimize both the noise and the energy of the signal, we need to select an antipodal signal pair that is frequency-limited to \([a, b]\) and has energy \( E \).