ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

Handout 12 Principles of Digital Communications
Problem Set 6 Mar. 29, 2023

PROBLEM 1. You are taking a multiple choice exam. Question number 5 allows for two possible answers. According to your first impression, answer 1 is correct with probability $\frac{1}{4}$ and answer 2 is correct with probability $\frac{3}{4}$. You would like to maximize your chance of giving the correct answer and you decide to have a look at what your neighbors on the left and right have to say. The neighbor on the left has answered $\hat{H}_L = 1$. He is an excellent student who has a record of being correct 90% of the time when asked a binary question. The neighbor on the right has answered $\hat{H}_R = 2$. He is a weaker student who is correct 70% of the time.

(a) You decide to use your first impression as a prior and to consider $\hat{H}_L$ and $\hat{H}_R$ as observations. Formulate the decision problem as a hypothesis testing problem.

(b) What is your answer $\hat{H}$?

PROBLEM 2. Two hypotheses $H = 0$ and $H = 1$ occur with probabilities $P_H(0) = p_0$ and $P_H(1) = 1 - p_0$, respectively. The observable Y takes values in the set of non-negative integers. Under hypothesis $H = 0$, Y is distributed according to a Poisson law with parameter λ_0, i.e.,

$$P_{Y|H}(y|0) = \frac{\lambda_0^y}{y!} e^{-\lambda_0}$$

Under hypothesis $H = 1$,

$$P_{Y|H}(y|1) = \frac{\lambda_1^y}{y!} e^{-\lambda_1}$$

This is a model for the reception of photons in optical communication.

(a) Derive the MAP decision rule by indicating likelihood and log-likelihood ratios.
Hint: The direction of an inequality changes if both sides are multiplied by a negative number.

(b) Derive an expression for the probability of error of the MAP decision rule.

(c) For $p_0 = \frac{1}{3}$, $\lambda_0 = 2$ and $\lambda_1 = 10$, compute (using a computer) the probability of error of the MAP decision rule.

(d) Repeat (c) with $\lambda_1 = 20$ and comment.

PROBLEM 3. Consider the m-ary hypothesis testing problem where the hypothesis is $H \in \{0, 1, \ldots, m-1\}$, and the observable is Y.

(a) Suppose under hypothesis $H = i$, $Y = (Y_1, \ldots, Y_n)$ is an i.i.d. sequence of Poisson random variables with parameter $\lambda_i > 0$. That is,

$$P_{Y_k|H}(y_k|i) = \frac{\lambda_i^{y_k}}{(y_k)!} e^{-\lambda_i}, \qquad y_k \in \{0, 1, 2, \ldots\}$$

Show that the *sample mean*

$$T(y_1, \ldots, y_n) = \frac{1}{n} \sum_{i=1}^{n} y_i,$$

is a sufficient statistic.

(b) Suppose under hypothesis $H = i$ the observable $Y = (Y_1, \ldots, Y_n)$ is described as

$$Y_k = \theta_i + Z_k, \qquad k = 1, 2, \ldots, n,$$

where $Z_k,\ k = 1, 2, \ldots, n$ are i.i.d. Exponential random variables with rate $\lambda_i > 0$, i.e.,

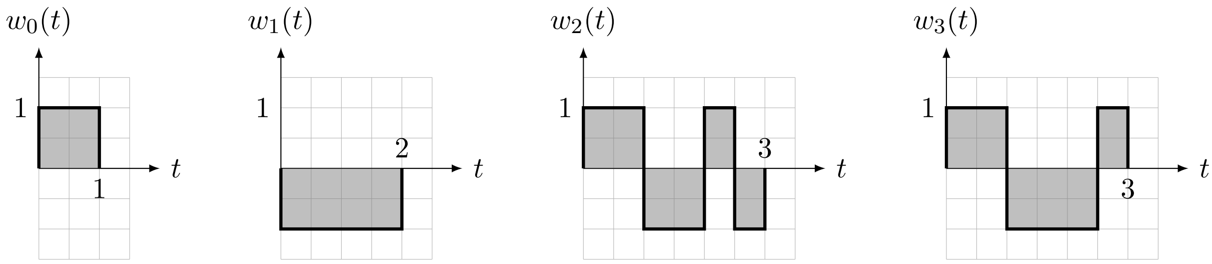
$$f_{Z_k|H}(z_k) = \begin{cases} \lambda_i e^{-\lambda_i z_k} & \text{if } z_k \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Show that the two-dimensional vector

$$T(y_1, \ldots, y_n) = \left(\min\{y_1, y_2, \ldots, y_n\}, \frac{1}{n} \sum_{k=1}^{n} y_k \right)$$

is a sufficient statistic.

PROBLEM 4. Consider the following signal set:



(a) Use the Gram–Schmidt (GS) procedure to find an orthonormal basis $\psi_1(t), \ldots, \psi_n(t)$.
Hint: No need to work out the intermediary steps of the GS procedure. The purpose of this exercise is to check, with hardly any calculation, your understanding of what the GS procedure does.

(b) Find the codeword $c_i \in \mathbb{R}^n$ that describes $w_i(t)$ with respect to your orthonormal basis. (No calculation needed.)

PROBLEM 5. Let a channel output be

$$R(t) = cXw(t) + N(t), \tag{1}$$

where $c > 0$ is some deterministic constant, X is a uniformly distributed random variable that takes values in $\{-3, -1, 1, 3\}$, $w(t) = \mathbb{1}_{[0,1)}(t)$, and $N(t)$ is white Gaussian noise of power spectral density $\frac{N_0}{2}$.

(a) Describe the receiver that, based on the channel output $R(t)$, decides on the value of X with least probability of error.

(b) Find the error probability of the receiver in part (a).

(c) Suppose now that you still use the receiver in part (a), but that the received signal is actually

$$R(t) = \frac{3}{4} cXw(t) + N(t),$$

i.e. you were unaware that the channel was attenuating the signal. What is the probability of error now?

(d) Suppose now that you still use the receiver in part (a) and that $R(t)$ is according to (1), but that the noise is colored. In fact, $N(t)$ is a zero-mean stationary Gaussian noise process of auto-covariance function

$$K_N(\tau) = \frac{1}{4\alpha} e^{-|\tau|/\alpha},$$

where $0 < \alpha < \infty$ is some deterministic real parameter. What is the probability of error now?