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Handout 11 Principles of Digital Communications
Solutions to Problem Set 5 Mar. 24, 2023

SOLUTION 1.

(a) At first look it may seem that the probability is uniformly distributed over the disk, but in the next part we will show that this is not true.

(b) We know that R is uniformly distributed in $[0,1]$ and Φ is uniformly distributed in $[0,2\pi)$, so we have $f_R(r) = 1$ if $0 \le r \le 1$ and $f_\Phi(\phi) = \frac{1}{2\pi}$ if $0 \le \phi < 2\pi$.

As these two random variables are independent, we have

$$f_{R,\Phi}(r,\phi) = \begin{cases} \frac{1}{2\pi} & 0 \le r \le 1 \text{ and } 0 \le \phi < 2\pi \\ 0 & \text{otherwise.} \end{cases}$$

It can be easily shown that the Jacobian determinant is $\det J = r = \sqrt{x^2+y^2}$. Therefore, the probability distribution in cartesian coordinates is

$$\begin{aligned} f_{X,Y}(x,y) &= \frac{1}{|\det J|} f_{R,\Phi}(r,\phi) \\ &= \begin{cases} \frac{1}{2\pi\sqrt{x^2+y^2}} & x^2+y^2 \le 1 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

(c) We see that the probability distribution is not distributed uniformly. This makes sense because rings of equal width have the same probability but not the same area.

SOLUTION 2.

(a) Let the two hypotheses be $H = 0$ and $H = 1$ when c_0 and c_1 are transmitted, respectively. The ML decision rule is

$$f_{Y_1Y_2|H}(y_1,y_2|1) \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} f_{Y_1Y_2|H}(y_1,y_2|0).$$

Because Z_1 and Z_2 are independent, we can write

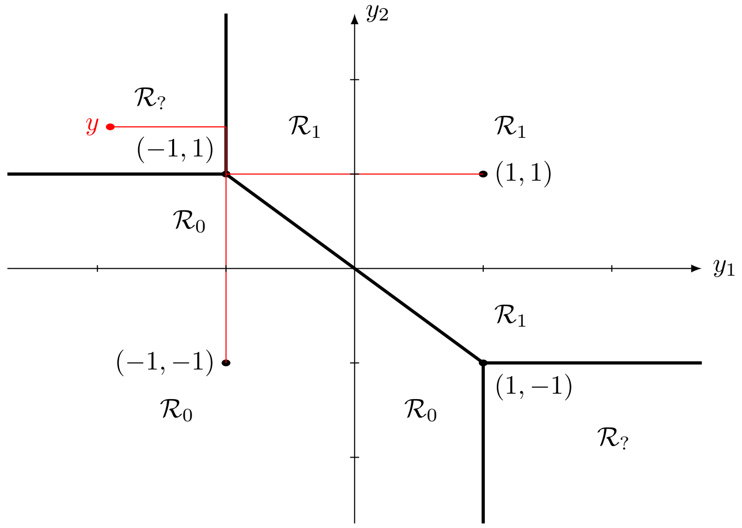
$$\frac{1}{2}e^{-|y_1-1|}\frac{1}{2}e^{-|y_2-1|} \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} \frac{1}{2}e^{-|y_1+1|}\frac{1}{2}e^{-|y_2+1|},$$

and, after taking the logarithm,

$$|y_1+1| + |y_2+1| \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} |y_1-1| + |y_2-1|.$$

(b) Because the hypotheses are equally likely and Z_1 and Z_2 have the same distribution, the decision region for $\hat{H} = 0$ contains the points closer to $(-1, -1)$ and the decision region for $\hat{H} = 1$ contains the points closer to $(1, 1)$. For this problem, the distance between the points (y_{11}, y_{12}) and (y_{21}, y_{22}) is the Manhattan distance, $|y_{11} - y_{21}| + |y_{12} - y_{22}|$, and not the Euclidian distance.

Let us first consider the points above the line $y_2 = -y_1$ in the figure below. It is easy to notice that the points in the positive quadrant are closer to $(1, 1)$ than to $(-1, -1)$, therefore they belong to $\mathcal{R}_1$ ($\hat{H} = 1$). This is also true if $\{(y_1 \geq -y_2) \cap (y_2 \in (-1, 0))\}$, or if $\{(y_2 \geq -y_1) \cap (y_1 \in (-1, 0))\}$.



Similar reasoning can be applied to the points below the diagonal to determine $\mathcal{R}_0$.

The points for which $\{(y_1 \leq -1) \cap (y_2 \geq 1)\}$ or $\{(y_1 \geq 1) \cap (y_2 \leq -1)\}$ are equally distanced to $(-1, -1)$ and $(1, 1)$, therefore they can belong to either $\mathcal{R}_0$ or $\mathcal{R}_1$ with the same probability. This region is named $\mathcal{R}_?$.

(c) The two hypotheses are equally probable for the region $\mathcal{R}_?$. Therefore, we can split this region in any way between the decision regions and have the same error probability. Because $\mathcal{R}_1$ is included in the region for which $y_2 > -y_1$ and $\mathcal{R}_0$ does not intersect the region for which $y_2 > -y_1$, the error probability is minimized by deciding $\hat{H} = 1$ if $(y_1 + y_2) > 0$.

(d)

$$\begin{aligned} P_e(0) &= \Pr\{Y_1 + Y_2 > 0 | H = 0\} \\ &= \Pr\{Z_1 + Z_2 - 2 > 0\} \\ &= \int_2^\infty \frac{e^{-w}}{4}(1 + w)\, dw \\ &= \frac{-e^{-w}}{4}(w + 2)\Big|_2^\infty = e^{-2}. \end{aligned}$$

By symmetry, and considering that the messages are equally likely, $P_e(0) = P_e(1) = P_e$.

Solution 3.

(a) The third component of c_i is zero for all i. Furthermore Z_1, Z_2 and Z_3 are zero mean i.i.d. Gaussian random variables. Hence,

$$f_{Y|H}(y|i) = f_{Z_1}(y_1 - c_{i,1}) f_{Z_2}(y_2 - c_{i,2}) f_{Z_3}(y_3),$$

which is in the form $g_i(T(y))h(y)$ for $T(y) = (y_1, y_2)^\mathsf{T}$ and $h(y) = f_{Z_3}(y_3)$. Hence, by the Fisher–Neyman factorization theorem, $T(Y) = (Y_1, Y_2)^\mathsf{T}$ is a sufficient statistic.

(b) We have $Y_3 = Z_3 = Z_2$. By observing Y_3, we can remove the noise in the second component of Y. Specifically, we have $c_{i,2} = Y_2 - Y_3$. If the second component is different for each hypothesis, then the receiver can make an error-free decision which is not possible using only $(Y_1, Y_2)^\mathsf{T}$ (see the next question for more on this). We can see that Y_3 contains very useful information and can't be discarded. Therefore, $(Y_1, Y_2)^\mathsf{T}$ is not a sufficient statistic.

(c) If we have only $(Y_1, Y_2)^\mathsf{T}$ then the hypothesis testing problem will be

$$H = i : (Y_1, Y_2) = (c_{i,1}, c_{i,2}) + (Z_1, Z_2) \quad i = \{0, 1\}$$

Using the fact that $c_0 = (1, 0, 0)^\mathsf{T}$ and $c_1 = (0, 1, 0)^\mathsf{T}$, the ML test becomes

$$y_1 - y_2 \underset{\hat{H}=1}{\overset{\hat{H}=0}{\gtrless}} 0$$

Under $H = 0$, $Y_1 - Y_2$ is a Gaussian random variable with mean 1 and variance $2\sigma^2$, and so $P_e(0) = Q(\frac{1}{\sqrt{2}\sigma})$. By symmetry $P_e(1) = Q(\frac{1}{\sqrt{2}\sigma})$, and so the error probability will be $P_e = \frac{1}{2}(P_e(0) + P_e(1)) = Q(\frac{1}{\sqrt{2}\sigma})$.

Now assume that we have access to Y_1, Y_2 and Y_3. Y_3 contains $Z_3 = Z_2$ under both hypotheses. Hence, $Y_2 - Y_3 = c_{i,2} + Z_2 - Z_3 = c_{i,2}$. This shows that at the receiver we can observe the second component of c_i without noise. As the second component is different under both hypotheses, we can make an error-free decision about H and the decision rule will be:

$$\hat{H} = \begin{cases} 0 & y_2 - y_3 = 0 \\ 1 & y_2 - y_3 = 1 \end{cases}$$

Clearly this decision rule minimizes the error probability. This shows once again that $(Y_1, Y_2)^\mathsf{T}$ can't be a sufficient statistic.

Solution 4.

(a) We use the Gram-Schmidt procedure:

1) The first step is to normalize the function $\beta_0(t)$, i.e. the first function of the basis that we are looking for is

$$\psi_0(t) = \frac{\beta_0(t)}{||\beta_0(t)||} = \frac{\beta_0(t)}{\sqrt{\int \beta_0(t)^2 \, dt}}$$

$$= \frac{\beta_0(t)}{\sqrt{\int_0^1 4t^2 \, dt}} = \frac{\sqrt{3}}{2}\beta_0(t) = \begin{cases} 0 & \text{if } t < 0 \\ \sqrt{3}t & \text{if } 0 \le t \le 1 \\ 0 & \text{if } t > 1 \end{cases}$$

2) Next, we subtract from $\beta_1(t)$ the components that are in the span of the currently established part of the basis, i.e. in the span of $\{\psi_0(t)\}$. This can be achieved by projecting $\beta_1(t)$ onto $\psi_0(t)$ and then subtracting this projection from $\beta_1(t)$, i.e.

$$\begin{aligned}
\alpha_1(t) &= \beta_1(t) - \langle \beta_1(t), \psi_0(t) \rangle \psi_0(t) = \beta_1(t) - \left(\int \beta_1(t)\psi_0(t)\ dt \right) \psi_0(t) \\
&= \beta_1(t) - \left(\frac{\sqrt{3}}{2} \right) \left(\frac{4}{3} \right) \psi_0(t) \\
&= \beta_1(t) - \frac{2}{\sqrt{3}} \psi_0(t) \\
&= \beta_1(t) - \beta_0(t).
\end{aligned}$$

From this, we find the second basis element as

$$\psi_1(t) = \frac{\alpha_1(t)}{||\alpha_1(t)||} = \begin{cases} 0 & \text{if } t < 1 \\ -\sqrt{3}(t-2) & \text{if } 1 \leq t \leq 2 \\ 0 & \text{if } t > 2 \end{cases}$$

3) Again, we subtract from $\beta_2(t)$ the components that are in the span of the currently established part of the basis, i.e. in the span of $\{\psi_0(t), \psi_1(t)\}$. This can be achieved by projecting $\beta_2(t)$ onto $\psi_0(t)$ and $\psi_1(t)$ and then subtracting both these projections from $\beta_2(t)$. For this step, it is *essential* that the basis elements $\{\psi_0(t), \psi_1(t)\}$ be orthonormal. Continuing the derivation, we obtain

$$\begin{aligned}
\alpha_2(t) &= \beta_2(t) - \langle \beta_2(t), \psi_0(t) \rangle \psi_0(t) - \langle \beta_2(t), \psi_1(t) \rangle \psi_1(t) \\
&= \beta_2(t) - \left(\int \beta_2(t)\psi_0(t)\ dt \right) \psi_0(t) - \left(\int \beta_2(t)\psi_1(t)\ dt \right) \psi_1(t) \\
&= \beta_2(t) - 0 - \alpha_1(t) \\
&= \beta_2(t) - \beta_0(t) + \beta_1(t),
\end{aligned}$$

and from this, we find the third basis element as

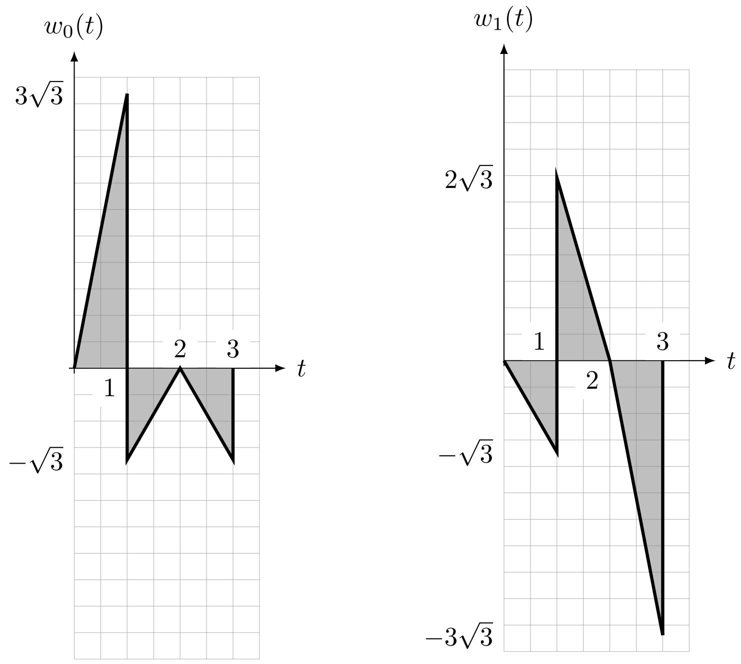
$$\psi_2(t) = \frac{\alpha_2(t)}{||\alpha_2(t)||} = \begin{cases} 0 & \text{if } t < 2 \\ -\sqrt{3}(t-2) & \text{if } 2 \leq t \leq 3 \\ 0 & \text{if } t > 3 \end{cases}$$

(b) By definition we can write $w_0(t)$ and $w_1(t)$ as follows

$$w_0(t) = 3\psi_0(t) - \psi_1(t) + \psi_2(t) = \begin{cases} 3\sqrt{3}t & \text{if } 0 \leq t < 1 \\ \sqrt{3}(t-2) & \text{if } 1 < t < 2 \\ -\sqrt{3}(t-2) & \text{if } 2 < t \leq 3 \end{cases}$$

and

$$w_1(t) = -\psi_0(t) + 2\psi_1(t) + 3\psi_2(t) = \begin{cases} -\sqrt{3}t & \text{if } 0 \leq t < 1 \\ -2\sqrt{3}(t-2) & \text{if } 1 < t < 2 \\ -3\sqrt{3}(t-2) & \text{if } 2 < t \leq 3 \end{cases}$$



(c)

$$\langle c_0, c_1 \rangle = -3 \cdot 1 - 1 \cdot 2 + 1 \cdot 3 = -2.$$

We know that $w_0(t)$ and $w_1(t)$ are both real, thus

$$\begin{aligned}\langle w_0(t), w_1(t) \rangle &= \int w_0(t) w_1(t) \ dt = \int_0^1 -9t^2 \ dt + \int_1^2 -6(t-2)^2 \ dt + \int_2^3 9(t-2)^2 \ dt \\ &= -\int_1^2 6(t-2)^2 \ dt = -2.\end{aligned}$$

We see that the inner products are equal as expected.

(d)

$$\begin{aligned}\|c_0\| &= \sqrt{\langle c_0, c_0 \rangle} = \sqrt{11}, \\ \|w_0\|^2 &= \int |w_0(t)|^2 \ dt = \int_0^1 27t^2 \ dt + \int_1^3 3(t-2)^2 \ dt = 9 + 2 = 11.\end{aligned}$$

We see that the norms are also equal.

Solution 5.

(a)

$$\|g_i\| = \sqrt{T}, \quad i = 1, 2, 3.$$

(b) Z_1 and Z_2 are independent since g_1 and g_2 are orthogonal. Hence Z is a Gaussian random vector $\sim \mathcal{N}(0, \sigma^2 I_2)$, where $\sigma^2 = \frac{N_0}{2}T$.

(c)

$$\begin{aligned}P_a &= \Pr\{Z_1 \in [1,2] \cap Z_2 \in [1,2]\} = \Pr\{Z_1 \in [1,2]\} \Pr\{Z_2 \in [1,2]\} \\ &= \left[Q\left(\frac{1}{\sigma}\right) - Q\left(\frac{2}{\sigma}\right) \right]^2,\end{aligned}$$

where $\sigma^2 = \frac{N_0}{2}T$.

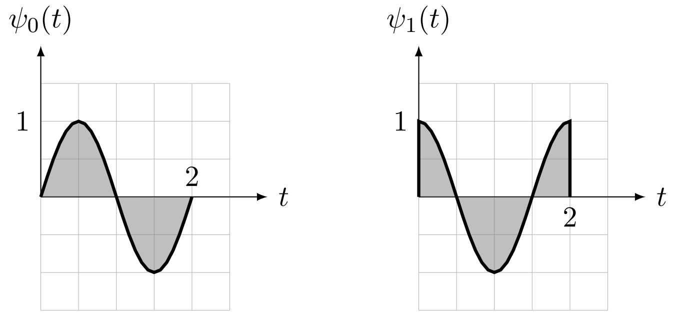
(d) $P_b = P_a$, since one obtains the square (b) from the square (a) via a rotation.

(e) $Z_3 = -Z_1$. $U = Z_1(1,-1)^\mathsf{T}$, and thus U can never be in (a), hence $Q_a = 0$.

(f) U is in square (c) if and only if $Z_1 \in [1,2]$. Hence $Q_c = Q\left(\frac{1}{\sigma}\right) - Q\left(\frac{2}{\sigma}\right)$, where $\sigma^2 = \frac{N_0}{2}T$.

Solution 6.

(a) An orthonormal basis for the signal space spanned by the waveforms is[1]:



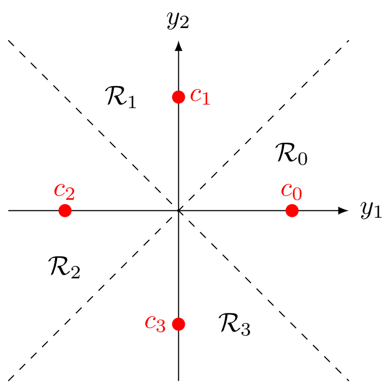
(b) The codewords representing the waveforms are

$$
\begin{aligned}
c_0 &= (\sqrt{\mathcal{E}}, 0) \\
c_1 &= (0, \sqrt{\mathcal{E}}) \\
c_2 &= (-\sqrt{\mathcal{E}}, 0) \\
c_3 &= (0, -\sqrt{\mathcal{E}})
\end{aligned}
$$

(c) As we have seen in the lecture, if $R(t)$ is the noisy received waveform, $(Y_0, Y_1) = (\langle R, \psi_0 \rangle, \langle R, \psi_1 \rangle)$ is a sufficient statistic for decision. Hence, we have the following hypothesis testing problem: Under $H = i$, $i = 0, 1, 2, 3$,

$$Y_i = c_i + Z,$$

where $Z \sim \mathcal{N}(0, \frac{N_0}{2} I_2)$. One can check that c_i, $i = 0, 1, 2, 3$ represent the QPSK codewords, and the decision regions for the ML receiver will be as follows:



[1] this can be obtained using the Gram-Schmidt procedure or simply by looking at the waveforms.

The distance between two adjacent codewords (say c_0 and c_1) is $d = \sqrt{2\mathcal{E}}$ and the error probability of the receiver is

$$
\begin{aligned}
P_e &= 2Q\left(\frac{d}{2\sigma}\right) - Q^2\left(\frac{d}{2\sigma}\right) \\
&= 2Q\left(\frac{\sqrt{2\mathcal{E}}}{2\sqrt{N_0/2}}\right) - Q^2\left(\frac{\sqrt{2\mathcal{E}}}{2\sqrt{N_0/2}}\right) \\
&= 2Q\left(\sqrt{\frac{\mathcal{E}}{N_0}}\right) - Q^2\left(\sqrt{\frac{\mathcal{E}}{N_0}}\right).
\end{aligned}
$$