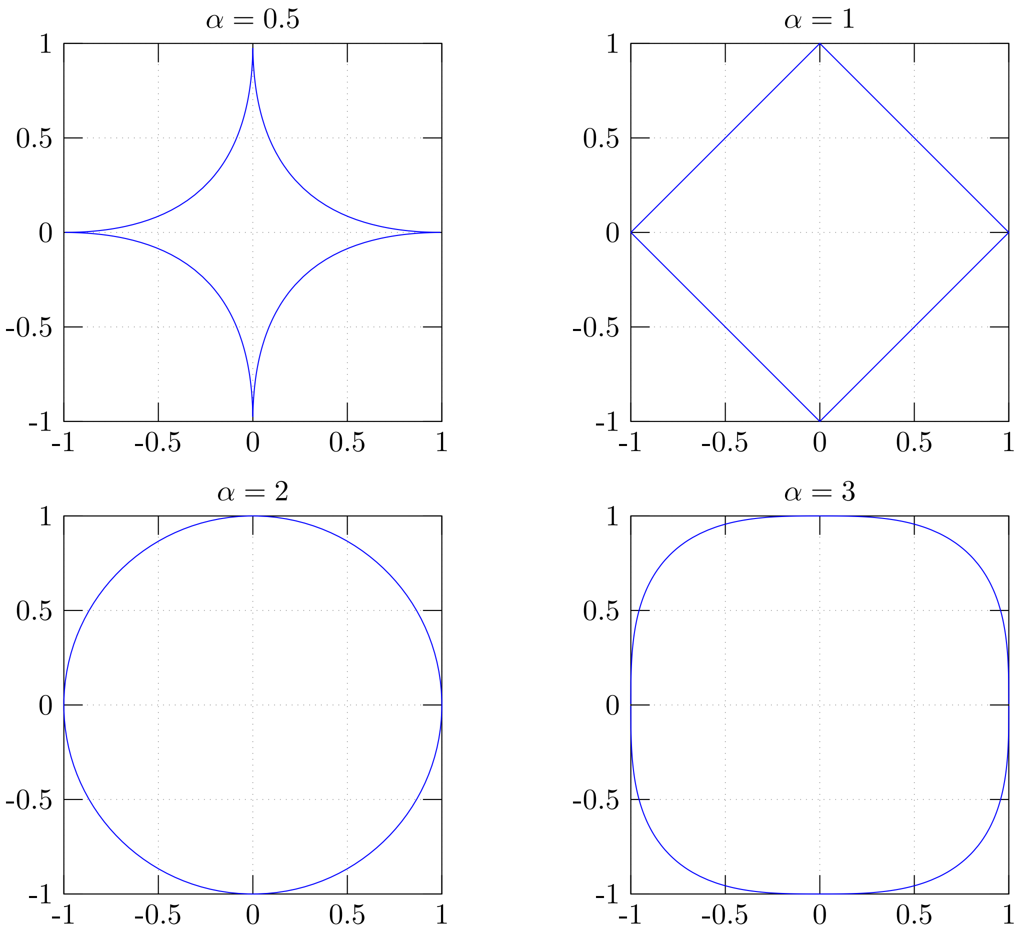
ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

Handout 29 | Principles of Digital Communications
Solutions to Problem Set 12 | May 26, 2023

SOLUTION 1.

(a) (i) The plots are shown below:



(ii) The joint density function is invariant under rotation for $\alpha = 2$ only. For this value of α, we have $X, Y \sim \mathcal{N}\left(0, \frac{1}{2}\right)$.

(b) (i) We know that we can write (x, y) in polar coordinates (r, θ). Hence in general the joint distribution of X and Y is a function of r and θ. Because of circular symmetry the joint distribution should not depend on θ, which means that $f_{X,Y}(x, y)$ can be written as a function of r. Hence if we denote this function by ψ and use the independence of X and Y, we have $f_X(x)f_Y(y) = \psi(r)$.

(ii) Taking the partial derivative with respect to x and using the chain rule for differentiation, we have $f_X'(x)f_Y(y) = \psi'(r)\frac{\partial r}{\partial x} = \psi'(r)\frac{x}{r}$. If we divide both sides by $xf_X(x)f_Y(y)$ we have $\frac{f_X'(x)}{xf_X(x)} = \frac{\psi'(r)}{r\psi(r)}$. Proceeding similarly for y, we obtain

$$\frac{f_X'(x)}{xf_X(x)} = \frac{\psi'(r)}{r\psi(r)} = \frac{f_Y'(y)}{yf_Y(y)}.$$

(iii) $\frac{f'_X(x)}{xf_X(x)}$ is a function of x while $\frac{f'_Y(y)}{yf_Y(y)}$ is a function of y. Hence the only way for the equality to hold is that both of them equal a constant. If we denote this constant by $-\frac{1}{\sigma^2}$, we reach the final result.

(iv) We have $\frac{f'_X(x)}{f_X(x)} = -\frac{x}{\sigma^2}$. Integrating both sides we have $\log(\frac{f_X(x)}{C}) = -\frac{x^2}{2\sigma^2}$. Hence $f_X(x) = C\exp(-\frac{x^2}{2\sigma^2})$. $f_X(x)$ is a probability density function and so should integrate to 1, which gives $C = \frac{1}{\sqrt{2\pi\sigma^2}}$. Hence $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp(-\frac{x^2}{2\sigma^2})$ and by symmetry $f_Y(y) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp(-\frac{y^2}{2\sigma^2})$, which shows that X and Y are Gaussian random variables.

Solution 2.

(a) Let $x_E(t) = x_R(t) + \mathrm{j}x_I(t)$. Then

$$\begin{aligned} x(t) &= \sqrt{2}\Re\{x_E(t)e^{\mathrm{j}2\pi f_c t}\} \\ &= \sqrt{2}\Re\{[x_R(t) + \mathrm{j}x_I(t)]e^{\mathrm{j}2\pi f_c t}\} \\ &= \sqrt{2}[x_R(t)\cos(2\pi f_c t) - x_I(t)\sin(2\pi f_c t)]. \end{aligned}$$

Hence, we have

$$x_{EI}(t) = \sqrt{2}\Re\{x_E(t)\}$$

and

$$x_{EQ}(t) = \sqrt{2}\Im\{x_E(t)\}.$$

(b) Let $x_E(t) = \alpha(t)e^{\mathrm{j}\beta(t)}$. Then

$$\begin{aligned} x(t) &= \sqrt{2}\Re\{x_E(t)e^{\mathrm{j}2\pi f_c t}\} \\ &= \sqrt{2}\Re\{\alpha(t)e^{\mathrm{j}\beta(t)}e^{\mathrm{j}2\pi f_c t}\} \\ &= \sqrt{2}\Re\{\alpha(t)e^{\mathrm{j}(2\pi f_c t+\beta(t))}\} \\ &= \sqrt{2}\alpha(t)\cos[2\pi f_c t + \beta(t)]. \end{aligned}$$

We thus have

$$x_E(t) = \alpha(t)e^{\mathrm{j}\beta(t)} = \frac{a(t)}{\sqrt{2}}e^{\mathrm{j}\theta(t)}.$$

(c) From (b) we see that

$$x_E(t) = \frac{A(t)}{\sqrt{2}}e^{\mathrm{j}\varphi}.$$

This is consistent with Example 7.9 (DSB-SC) given in the text. We can also verify:

$$\begin{aligned} x(t) &= \sqrt{2}\Re\{x_E(t)e^{\mathrm{j}2\pi f_c t}\} \\ &= \sqrt{2}\Re\left\{\frac{A(t)}{\sqrt{2}}e^{\mathrm{j}\varphi}e^{\mathrm{j}2\pi f_c t}\right\} \\ &= \Re\{A(t)e^{\mathrm{j}(2\pi f_c t+\varphi)}\} \\ &= A(t)\cos(2\pi f_c t + \varphi). \end{aligned}$$

SOLUTION 3.

(a) The key observation is that while $e^{j2\pi f_1 t}$ and $e^{-j2\pi f_1 t}$ are two different signals if $f_1 \neq 0$, $\Re\{e^{j2\pi f_1 t}\}$ and $\Re\{e^{-j2\pi f_1 t}\}$ are identical.

Therefore, if we fix $f_1 \neq 0$ and choose $a_1(t)$ and $a_2(t)$ so that $a_1(t)e^{j2\pi f_c t} = e^{j2\pi f_1 t}$ and $a_2(t)e^{j2\pi f_c t} = e^{-j2\pi f_1 t}$, we get $a_1(t) \neq a_2(t)$ and $\Re\left\{a_1(t)e^{j2\pi f_c t}\right\} = \Re\left\{a_2(t)e^{j2\pi f_c t}\right\}$.

Let $a_1(t) = e^{-j2\pi(f_c - f_1)t}$ and $a_2(t) = e^{-j2\pi(f_c+f_1)t}$. Then $a_1(t) \neq a_2(t)$ and

$$\sqrt{2}\Re\left\{a_1(t)e^{j2\pi f_c t}\right\} = \sqrt{2}\Re\left\{a_2(t)e^{j2\pi f_c t}\right\}.$$

(b) Let $b(t) = a(t)e^{j2\pi f_c t}$, which represents a translation of $a(t)$ in the frequency domain. If $a_{\mathcal{F}}(f) = 0$ for $f < -f_c$, then $b_{\mathcal{F}}(f) = 0$ for $f < 0$. Because $\Re\{b(t)\} = \frac{1}{2}\left(a(t)e^{j2\pi f_c t} + a^*(t)e^{-j2\pi f_c t}\right)$, taking the real part has a scaling effect and adds a negative-frequency component. The negative spectrum is canceled by the $h_>$ filter, and the scaling is compensated by the $\sqrt{2}$ factors from the up-converter and down-converter. Multiplying by $e^{-j2\pi f_c t}$ translates the spectrum back to the initial position. In conclusion, we obtain $a(t)$.

(c) Take any baseband signal $u(t)$ with frequency domain support $[-f_c-\Delta, f_c+\Delta]$, $\Delta > 0$. The signal can be real-valued or complex-valued (for example $u_{\mathcal{F}}(f) = \mathbb{1}_{[-f_c-\Delta, f_c+\Delta]}(f)$, which is a sinc in time domain). After we up-convert, the support of $u_{\mathcal{F}}(f)$ will not extend beyond $2f_c + \Delta$. When we chop the negative frequencies we obtain a support contained in $[0, 2f_c+\Delta]$ and when we shift back to the left the support will be contained in $[-f_c, f_c + \Delta]$, which is too small to be the support of $u_{\mathcal{F}}(f)$.

(d) In time domain:

$$\begin{aligned} w(t) &= \sqrt{2}\Re\{a(t)e^{j2\pi f_c t}\} \\ &\overset{a\in\mathbb{R}}{=} \sqrt{2}a(t)\cos(2\pi f_c t). \end{aligned}$$

Therefore,

$$a(t) = \frac{w(t)}{\sqrt{2}\cos(2\pi f_c t)}.$$

In frequency domain: If $a_{\mathcal{F}}(f) = 0$ for $f < -f_c$, we obtain $a(t)$ as described in (b). In the following, we consider the case $a_{\mathcal{F}}(f) \neq 0$ for $f < -f_c$.

We have $w_{\mathcal{F}}(f) = \frac{1}{\sqrt{2}}\left[a_{\mathcal{F}}(f - f_c) + a_{\mathcal{F}}(f + f_c)\right] = a_{\mathcal{F}}^+(f) + a_{\mathcal{F}}^-(f)$, with $a_{\mathcal{F}}^+(f) = \frac{1}{\sqrt{2}}a_{\mathcal{F}}(f - f_c)$ and $a_{\mathcal{F}}^-(f) = \frac{1}{\sqrt{2}}a_{\mathcal{F}}(f + f_c)$, respectively. These two components have overlapping support in some interval centered at 0. However, there is no overlap for sufficiently large frequencies. This means that for sufficiently large frequencies f we have $w_{\mathcal{F}}(f) = \frac{1}{\sqrt{2}}a_{\mathcal{F}}^+(f)$, which implies that from $w_{\mathcal{F}}(f)$ we can observe the right tail of $a_{\mathcal{F}}^+(f)$ and use that information to remove the right tail of $a_{\mathcal{F}}^-(f)$ from $w_{\mathcal{F}}(f)$ (the right tails of $a_{\mathcal{F}}^+(f)$ and $a_{\mathcal{F}}^-(f)$ are the same because $a(t)$ is real). Hence, from $w_{\mathcal{F}}(f)$ we can read more of the right tail of $a_{\mathcal{F}}^+(f)$. The procedure can be repeated until we get to see $a_{\mathcal{F}}^+(f)$ for all frequencies above f_c. At this point, using $a_{\mathcal{F}}(f) = a_{\mathcal{F}}^+(f + f_c)\sqrt{2}$ and the fact that $a(t)$ is real-valued, we have $a_{\mathcal{F}}(f)$ for the positive frequencies, hence for all frequencies.

SOLUTION 4.

$$\begin{aligned}
x(t)\sqrt{2}\cos(2\pi f_c t) &= x(t)\left[\frac{e^{\mathrm{j}2\pi f_c t}+e^{-\mathrm{j}2\pi f_c t}}{\sqrt{2}}\right] \\
&= \sqrt{2}\Re\{x_E(t)e^{\mathrm{j}2\pi f_c t}\}\left[\frac{e^{\mathrm{j}2\pi f_c t}+e^{-\mathrm{j}2\pi f_c t}}{\sqrt{2}}\right] \\
&= \left[\frac{x_E(t)e^{\mathrm{j}2\pi f_c t}+x_E^*(t)e^{-\mathrm{j}2\pi f_c t}}{\sqrt{2}}\right]\left[\frac{e^{\mathrm{j}2\pi f_c t}+e^{-\mathrm{j}2\pi f_c t}}{\sqrt{2}}\right] \\
&= \frac{x_E(t)e^{\mathrm{j}4\pi f_c t}+x_E(t)+x_E^*(t)+x_E^*(t)e^{-\mathrm{j}4\pi f_c t}}{2}.
\end{aligned}$$

At the lowpass filter output we have

$$\frac{x_E(t)+x_E^*(t)}{2}=\Re\{x_E(t)\}.$$

The calculation for the other path is similar.

SOLUTION 5.

(a) Notice that the sinusoids of $w(t)$ have a period of $T_c = 4$ ms units of time, which implies that $f_c = \frac{1}{T_c} = \frac{1}{4\text{ ms}} = 250$ Hz.

(b) Notice that the phase of the sinusoidal signal changes every $T_s = 4$ ms. (Here we have $T_s = T_c$, but in general it is not the case. In practice we usually have $T_s \gg T_c$. See the note at the end.)

The expression of $w(t)$ as a function of t is:

$$\begin{aligned}
w(t) &= \begin{cases} 4\cos(2\pi f_c t-\frac{\pi}{2}) & t\in]0,T_s[\\ 4\cos(2\pi f_c t) & t\in]T_s,2T_s[\\ 4\cos(2\pi f_c t+\pi) & t\in]2T_s,3T_s[\\ 4\cos(2\pi f_c t+\frac{\pi}{2}) & t\in]3T_s,4T_s[\end{cases}
= \begin{cases} \Re\left\{4e^{\mathrm{j}(2\pi f_c t-\frac{\pi}{2})}\right\} & t\in]0,T_s[\\ \Re\left\{4e^{\mathrm{j}(2\pi f_c t)}\right\} & t\in]T_s,2T_s[\\ \Re\left\{4e^{\mathrm{j}(2\pi f_c t+\pi)}\right\} & t\in]2T_s,3T_s[\\ \Re\left\{4e^{\mathrm{j}(2\pi f_c t+\frac{\pi}{2})}\right\} & t\in]3T_s,4T_s[\end{cases} \\
&= \begin{cases} \Re\left\{-4\mathrm{j}e^{\mathrm{j}2\pi f_c t}\right\} & t\in]0,T_s[\\ \Re\left\{4e^{\mathrm{j}2\pi f_c t}\right\} & t\in]T_s,2T_s[\\ \Re\left\{-4e^{\mathrm{j}2\pi f_c t}\right\} & t\in]2T_s,3T_s[\\ \Re\left\{4\mathrm{j}e^{\mathrm{j}2\pi f_c t}\right\} & t\in]3T_s,4T_s[\end{cases}
= \sqrt{2}\Re\left\{w_E(t)e^{\mathrm{j}2\pi f_c t}\right\},
\end{aligned}$$

where

$$\begin{aligned}
w_E(t) &= -\frac{4\mathrm{j}}{\sqrt{2}}\mathbb{1}\{t\in]0,T_s[\} + \frac{4}{\sqrt{2}}\mathbb{1}\{t\in]T_s,2T_s[\} \\
&\quad -\frac{4}{\sqrt{2}}\mathbb{1}\{t\in]2T_s,3T_s[\} + \frac{4\mathrm{j}}{\sqrt{2}}\mathbb{1}\{t\in]3T_s,4T_s[\} \\
&= -\mathrm{j}\sqrt{8T_s}\frac{1}{\sqrt{T_s}}\mathbb{1}\{t\in]0,T_s[\} + \sqrt{8T_s}\frac{1}{\sqrt{T_s}}\mathbb{1}\{t\in]T_s,2T_s[\} \\
&\quad -\sqrt{8T_s}\frac{1}{\sqrt{T_s}}\mathbb{1}\{t\in]2T_s,3T_s[\} + \mathrm{j}\sqrt{8T_s}\frac{1}{\sqrt{T_s}}\mathbb{1}\{t\in]3T_s,4T_s[\}.
\end{aligned}$$

If we define $\psi(t) = \frac{1}{\sqrt{T_s}}\mathbb{1}\{t \in]0, T_s[\}$, $c_0 = -\mathtt{j}\sqrt{8T_s}$, $c_1 = \sqrt{8T_s}$, $c_2 = -\sqrt{8T_s}$ and $c_3 = \mathtt{j}\sqrt{8T_s}$, we get

$$w_E(t) = \sum_{i=0}^{3} c_i \psi(t - iT_s). \tag{1}$$

Therefore, the pulse used in the waveform former is $\psi(t) = \frac{1}{\sqrt{T_s}}\mathbb{1}\{t \in]0, T_s[\}$, and the waveform former output signal is given by (1). The orthonormal basis that is used is $\{\psi(t - iT_s)\}_{i=0}^{3}$.

(c) The symbol sequence is $\{c_0,\ c_1,\ c_2,\ c_3\} = \left\{-\mathtt{j}\sqrt{\mathcal{E}_s},\ \sqrt{\mathcal{E}_s},\ -\sqrt{\mathcal{E}_s},\ \mathtt{j}\sqrt{\mathcal{E}_s}\right\}$, where $\mathcal{E}_s = 8T_s$. We can see that the symbol alphabet is $\left\{\sqrt{\mathcal{E}_s},\ \mathtt{j}\sqrt{\mathcal{E}_s},\ -\sqrt{\mathcal{E}_s},\ -\mathtt{j}\sqrt{\mathcal{E}_s}\right\}$.

(d) We have:

- The output sequence of the encoder is the symbol sequence, which is

$$\{c_0,\ c_1,\ c_2,\ c_3\} = \left\{-\mathtt{j}\sqrt{\mathcal{E}_s},\ \sqrt{\mathcal{E}_s},\ -\sqrt{\mathcal{E}_s},\ \mathtt{j}\sqrt{\mathcal{E}_s}\right\}.$$

- The symbol alphabet contains 4 symbols. This means that each symbol represents two bits. Since the symbol rate is $f_s = \frac{1}{T_s} = 250$ symbols/s, the bit rate is $2 \times 250 = 500$ bits/s.
- The input/output mapping can be obtained by assigning two bits for each symbol in the symbol alphabet. Keeping in mind that it is better to minimize the number of bit-differences between close symbols, we obtain the following input/output mapping (which is not unique, i.e., we can obtain other mappings that satisfy the mentioned criterion): $\sqrt{\mathcal{E}_s} \longleftrightarrow 00$, $\mathtt{j}\sqrt{\mathcal{E}_s} \longleftrightarrow 01$, $-\sqrt{\mathcal{E}_s} \longleftrightarrow 11$ and $-\mathtt{j}\sqrt{\mathcal{E}_s} \longleftrightarrow 10$.
- Assuming that the above input/output mapping was used, we can obtain the input sequence of the encoder: 10001101.

Note that in this example, we have $T_s = T_c$, so $f_c = f_s$. This is very unusual. In practice we almost always have $f_c \gg f_s$, especially if we are using electromagnetic waves.