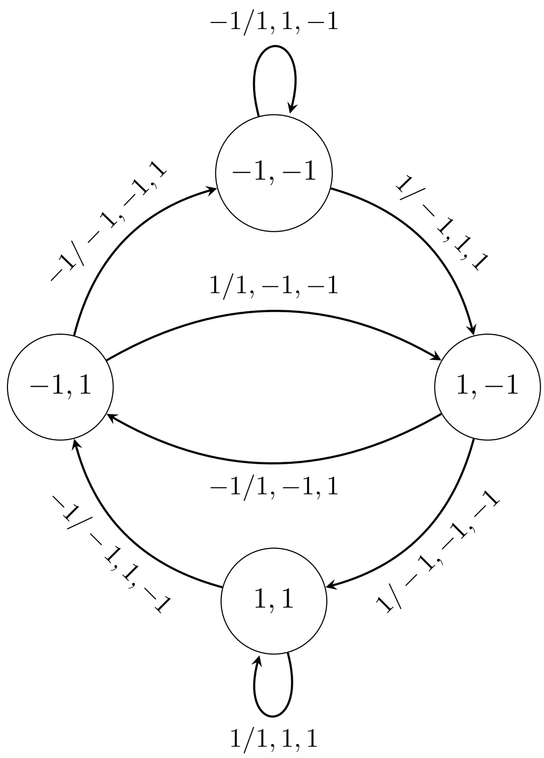
ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

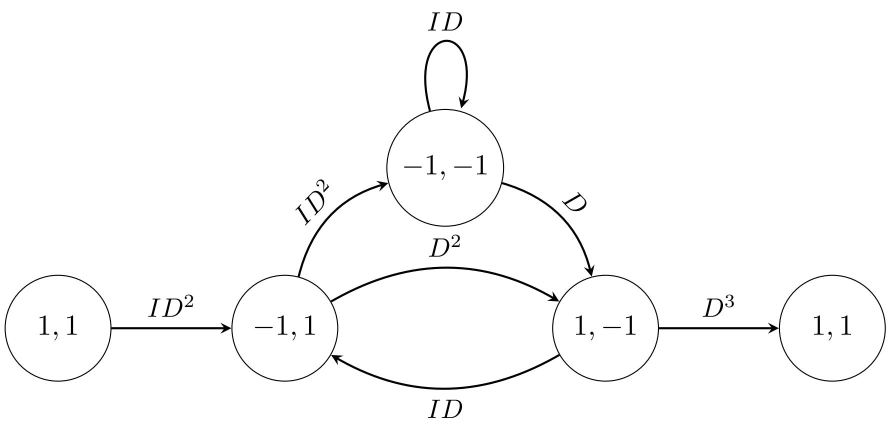
Handout 27 Principles of Digital Communications
Solutions to Problem Set 11 May 19, 2023

SOLUTION 1.

(a) The state diagram and detour flow graph are respectively shown below:



State diagram



Detour flow graph

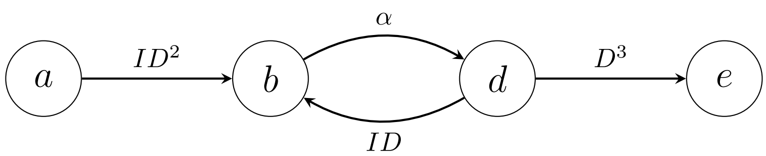
(b) Let a, b, c, d, e respectively represent the states $(1, 1)$, $(-1, 1)$, $(-1, -1)$, $(1, -1)$ and $(1, 1)$. We have

$$T_b = T_d ID + T_a ID^2$$
$$T_c = T_c ID + T_b ID^2$$
$$T_d = T_b D^2 + T_c D.$$

Substituting $T_c = T_b \frac{ID^2}{1-ID}$ in the third equation above,

$$\begin{aligned} T_d &= T_b D^2 + T_b \frac{ID^3}{1-ID} \\ &= T_b \left(D^2 + \frac{ID^3}{1-ID} \right) \\ &= T_b \frac{D^2}{1-ID} \\ &= T_b \alpha, \end{aligned}$$

with $\alpha = \frac{D^2}{1-ID}$. The detour flow graph can thus be simplified to:



In $T_b = T_d ID + T_a ID^2$, we substitute for T_d to get

$$T_b = T_a \frac{ID^2(1-ID)}{1-ID-ID^3}.$$

It follows that

$$T_d = T_b \frac{D^2}{1-ID} = T_a \frac{ID^4}{1-ID-ID^3},$$

and that

$$T(I,D) = T_e = T_a \frac{ID^7}{1-ID-ID^3}.$$

Taking the derivative yields

$$\frac{\partial T(I,D)}{\partial I} = \frac{D^7(1-ID-ID^3) - ID^7(-D-D^3)}{(1-ID-ID^3)^2} = \frac{D^7}{(1-ID-ID^3)^2}.$$

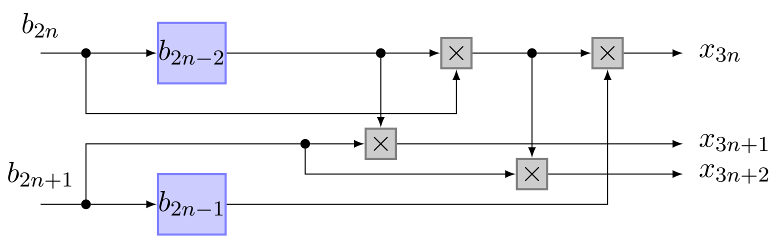
Therefore, we find

$$\begin{aligned} P_b &\leq \left. \frac{\partial T(I,D)}{\partial I} \right|_{I=1, D=z} \\ &= \frac{z^7}{(1-z-z^3)^2}, \end{aligned}$$

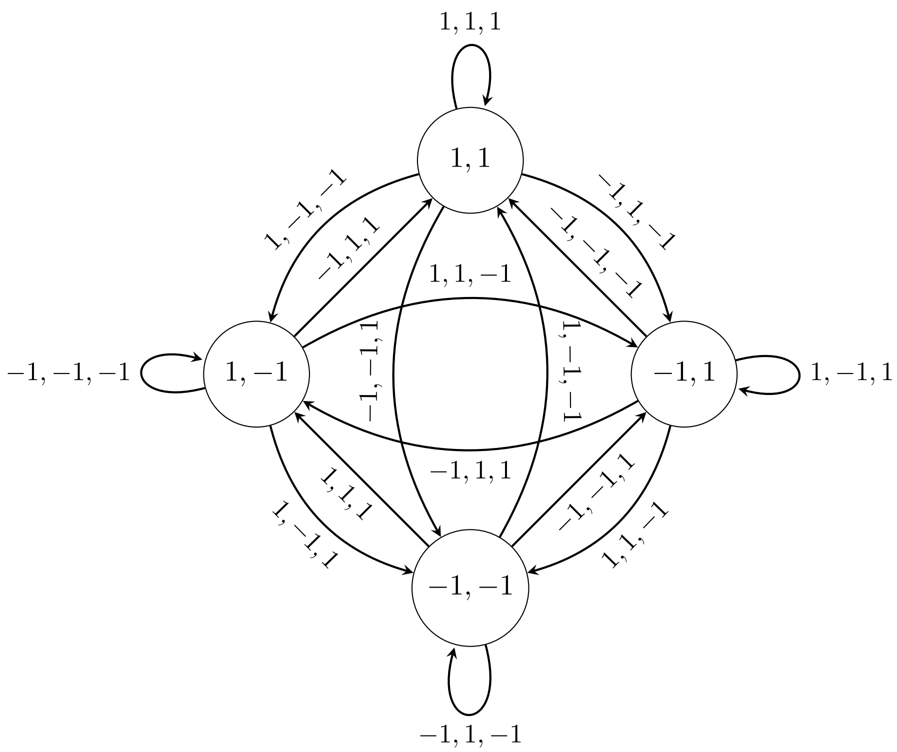
where $z = e^{-\frac{\mathcal{E}_s}{N_0}}$.

Solution 2.

(a) An implementation of the encoder will be as follows:



(b) The state diagram is shown below. We use the following terminology: the state label is x, y, where x is the "state of the even sub-sequence", i.e. contains b_{2n-2}, and y is the "state of the odd sub-sequence", i.e., contains b_{2n-1}. On the arrows, we only mark the outputs; the input required to make a particular transition is simply the next state, therefore we omitted it. Transitions are labeled with the value of $x_{3n}, x_{3n+1}, x_{3n+2}$.



(c) We use

$$P_b \leq \frac{1}{k_0} \frac{\partial T(I, D)}{\partial I} \bigg|_{I=1, D=z},$$

where $z = e^{-\frac{\mathcal{E}_s}{N_0}}$ and k_0 is the number of inputs per section of the trellis. In this problem, $k_0 = 2$. Since there are three channel symbols per two source symbols, we find that $\mathcal{E}_s = 2\mathcal{E}_b/3$.

From the state diagram we can derive the generating functions of the detour flow graph:

$$\begin{aligned}
T(I, D) &= D^3 T_{-1,1} + D^2 T_{-1,-1} + D T_{1,-1} \\
T_{1,-1} &= IDT_{-1,1} + IT_{-1,-1} + ID^3 T_{1,-1} + ID^2 T_{1,1} \\
T_{-1,-1} &= I^2 DT_{-1,1} + I^2 D^2 T_{-1,-1} + I^2 DT_{1,-1} + I^2 D^2 T_{1,1} \\
T_{-1,1} &= IDT_{-1,1} + ID^2 T_{-1,-1} + IDT_{1,-1} + ID^2 T_{1,1}.
\end{aligned}$$

Solving the system gives

$$T(I, D) = T_{1,1} \frac{D^2 I(D^6 I + D^5 I^2 - D^3 - D^4 I - D)}{-D^5 I^3 - D^4 I^2 + D^3 I + 2D^2 I^2 + D^2 I + DI^3 + DI^2 + DI - 1},$$

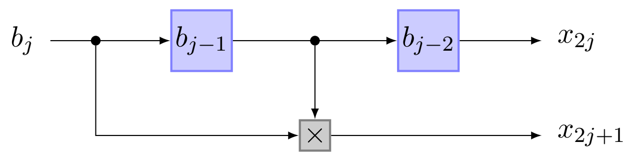
on which we can apply the formula above.

SOLUTION 3.

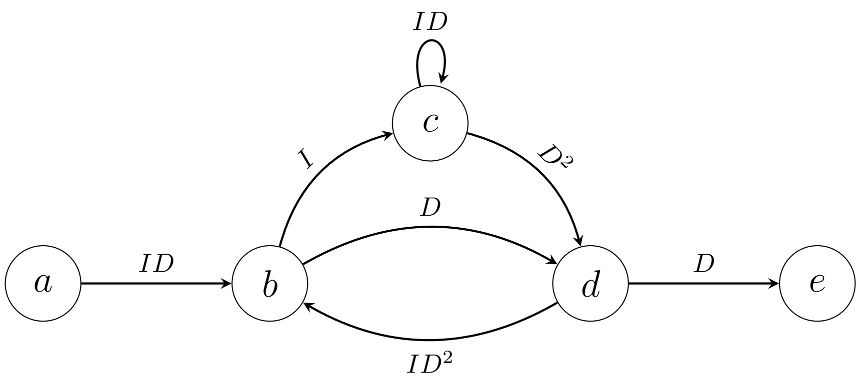
(a) Since the state is (b_{j-1}, b_{j-2}), we need two shift registers. From the finite state machine, we can derive a table that relates the state (b_{j-1}, b_{j-2}) and the current input b_j with the two outputs (x_{2j}, x_{2j+1}):

b_j	b_{j-1}	b_{j-2}	x_{2j}	x_{2j+1}
1	1	1	1	1
1	1	−1	−1	1
1	−1	1	1	−1
1	−1	−1	−1	−1
−1	1	1	1	−1
−1	1	−1	−1	−1
−1	−1	1	1	1
−1	−1	−1	−1	1

We can easily notice that the column of x_{2j} is the same as the column of b_{j-2}. Therefore, $x_{2j} = b_{j-2}$. On the other hand, we see that $x_{2j+1} = b_{j-1}$ if $b_j = 1$ and $x_{2j+1} = -b_{j-1}$ if $b_j = -1$. Therefore $x_{2j+1} = b_j \cdot b_{j-1}$, which gives us the following encoder.



(b) The detour flow graph (with respect to the all-one sequence) is given below:



We have

$$\begin{aligned} T_b &= T_a ID + T_d ID^2 \\ T_c &= T_b I + T_c ID \\ T_d &= T_c D^2 + T_b D \\ T_e &= T_d D \end{aligned}$$

The solution of this system is $T_e = T_a \frac{ID^3}{1-ID-ID^3}$. Hence,

$$\begin{aligned} P_b \le \left. \frac{\partial T(I,D)}{\partial I} \right|_{I=1,D=z} &= \left. \frac{D^3(1-ID-ID^3) + ID^3(D+D^3)}{(1-ID-ID^3)^2} \right|_{I=1,D=z} \\ &= \frac{z^3}{(1-z-z^3)^2}, \end{aligned}$$

where $z = e^{-\frac{\mathcal{E}_b}{2N_0}}$.

SOLUTION 4.

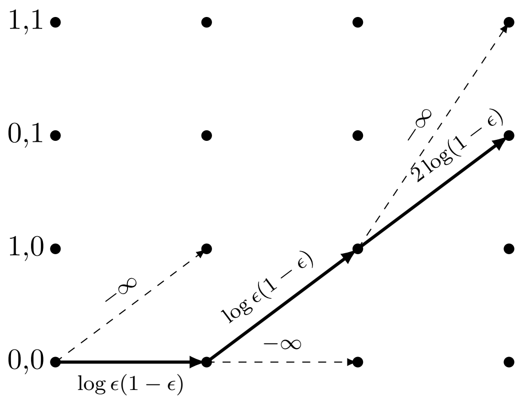
(a) The decoder is the same as in the example we have seen in Chapter 6 once the following isomorphic mapping is applied: $\{1 \to 0, -1 \to 1\}$. Figure 6.4 shows the trellis of the encoder.

(b) Given the observation $y = (y_1, \ldots, y_n)$, the ML codeword is given by $\arg\max_{x \in \mathcal{C}} p(y|x)$ where $\mathcal{C}$ represents the set of codewords (i.e., the set of all possible paths on the trellis). Alternately, the ML codeword is given by $\arg\max_{x \in \mathcal{C}} \sum_{i=1}^{n} \log p(y_i|x_i)$.

Hence, a branch metric for the BEC is

$$\log p(y_i|x_i) = \begin{cases} \log \epsilon & \text{if } y_i = ?, \\ \log(1-\epsilon) & \text{if } y_i = x_i, \\ -\infty & \text{if } y_i = 1 - x_i. \end{cases}$$

(c) Given the observation $(0, ?, ?, 1, 0, 1)$, one can compute the branch metric in the trellis. Note that we do not need to further elaborate paths with a $-\infty$ metric. The decoding results $(0, 1, 0)$.



(d) We refer to the example shown in Chapter 6, where we have the same encoder, but a different channel. We have seen that

$$P_b \leq \frac{z^5}{(1-2z)^2}.$$

To determine z we use the Bhattacharyya bound, which in our case is

$$z = \sum_{y \in \{0,1,?\}} \sqrt{P(y|1)P(y|0)} = \epsilon.$$

Thus we have the following bound:

$$P_b \leq \frac{\epsilon^5}{(1-2\epsilon)^2}.$$