PROBLEM 1. Random variables $X$ and $Y$ are correlated Gaussian variables:

$$
\begin{pmatrix}
X \\
Y
\end{pmatrix} \sim N_2 \left( \begin{pmatrix} 0 \\ 0 \end{pmatrix} ; K = \begin{bmatrix}
\sigma_x^2 & \rho \sigma_x \sigma_y \\
\rho \sigma_x \sigma_y & \sigma_y^2
\end{bmatrix} \right).
$$

Find $I(X;Y)$.

PROBLEM 2. Consider an additive noise channel with input $x \in \mathbb{R}$, and output

$$
Y = x + Z
$$

where $Z$ is a real random variable independent of the input $x$, has zero mean and variance equal to $\sigma^2$.

In this problem we prove in a different way from the lecture that the Gaussian channel has the smallest capacity among all additive noise channels of a given noise variance. Let $\mathcal{N}_{\sigma^2}$ denote the Gaussian density with zero mean and variance $\sigma^2$.

(a) Denote the input probability density by $p_X$. Verify that

$$
I(X;Y) = \int \int p_X(x)p_Z(y-x) \ln \frac{p_Z(y-x)}{p_Y(y)} \, dx \, dy \text{ nats.}
$$

where $p_Y$ is the density of the output when the input has density $p_X$.

(b) Now set $p_X = \mathcal{N}_P$. Verify that

$$
\frac{1}{2} \ln(1 + P/\sigma^2) = \int \int p_X(x)p_Z(y-x) \ln \frac{\mathcal{N}_{\sigma^2}(y-x)}{\mathcal{N}_{P+\sigma^2}(y)} \, dx \, dy.
$$

(c) Still with $p_X = \mathcal{N}_P$, show that

$$
\frac{1}{2} \ln(1 + P/\sigma^2) - I(X;Y) \leq 0.
$$

[Hint: use (a) and (b) and $\ln t \leq t - 1$.]

(d) Show that an additive noise channel with noise variance $\sigma^2$ and input power $P$ has capacity at least $\frac{1}{2} \log_2(1 + P/\sigma^2)$ bits per channel use. Conclude that the Gaussian channel has the smallest capacity among all additive noise channels of a given noise variance.

PROBLEM 3. A discrete memoryless channel has three input symbols: $\{-1, 0, 1\}$, and two output symbols: $\{1, -1\}$. The transition probabilities are

$$
p(-1|1) = p(1|1) = 1, \quad p(1|0) = p(-1|0) = 0.5.
$$

Find the capacity of this channel with cost constraint $\beta$, if the cost function is $b(x) = x^2$. 
Problem 4. Consider a vector Gaussian channel described as follows:

\[
\begin{align*}
Y_1 &= x + Z_1 \\
Y_2 &= Z_2
\end{align*}
\]

where \(x\) is the input to the channel constrained in power to \(P\); \(Z_1\) and \(Z_2\) are jointly Gaussian random variables with \(E[Z_1] = E[Z_2] = 0\), \(E[Z_1^2] = E[Z_2^2] = \sigma^2\) and \(E[Z_1 Z_2] = \rho \sigma^2\), with \(\rho \in [-1, 1]\), and independent of the channel input.

(a) Consider a receiver that discards \(Y_2\) and decodes the message based only on \(Y_1\). What rates are achievable with such a receiver?

(b) Consider a receiver that forms \(Y = Y_1 - \rho Y_2\), and decodes the message based only on \(Y\). What rates are achievable with such a receiver?

(c) Find the capacity of the channel and compare it to the part (b).

Problem 5. Suppose \((X_1, \ldots, X_n, Y_1, \ldots, Y_m)\) is an \(n + m\) dimensional Gaussian random vector with covariance matrix \(K\). Partition the \((n + m) \times (n + m)\) matrix \(K\) as

\[
K = \begin{bmatrix}
K_{11} & K_{12} \\
K_{21} & K_{22}
\end{bmatrix}
\]

where \(K_{11}\) is \(n \times n\) and \(K_{22}\) is \(m \times m\).

(a) Express \(h(X_1, \ldots, X_n)\), \(h(Y_1, \ldots, Y_n)\) and \(h(X_1, \ldots, X_n, Y_1, \ldots, Y_m)\) in terms of the matrices above.

(b) Show if the matrix \(A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\) is positive definite, then \(\det(A) \leq \det(A_{11}) \det(A_{22})\).

\[\text{[Hint: for any positive definite matrix } A, \ f(x) = \det(2\pi A)^{-1/2} \exp(-\frac{1}{2} x^T A^{-1} x) \text{ is a probability density.]}\]