Problem 1. Show that a cascade of $n$ identical binary symmetric channels,

$$X_0 \xrightarrow{\text{BSC #1}} X_1 \rightarrow \cdots \rightarrow X_{n-1} \xrightarrow{\text{BSC #n}} X_n$$

each with raw error probability $p$, is equivalent to a single BSC with error probability \( \frac{1}{2} \left(1 - (1 - 2p)^n\right) \) and hence that \( \lim_{n \to \infty} I(X_0; X_n) = 0 \) if $p \neq 0, 1$. Thus, if no processing is allowed at the intermediate terminals, the capacity of the cascade tends to zero.

Problem 2. Consider a memoryless channel with transition probability matrix $P_{Y|X}(y|x)$, with $x \in X$ and $y \in Y$. For a distribution $Q$ over $X$, let $I(Q)$ denote the mutual information between the input and the output of the channel when the input distribution is $Q$. Show that for any two distributions $Q$ and $Q'$ over $X$,

(a) $I(Q') \leq \sum_{x \in X} Q'(x) \sum_{y \in Y} P_{Y|X}(y|x) \log \left( \frac{P_{Y|X}(y|x)}{\sum_{x' \in X} P_{Y|X}(y|x')Q(x')} \right)$

(b) $C \leq \max_x \sum_{y \in Y} P_{Y|X}(y|x) \log \left( \frac{P_{Y|X}(y|x)}{\sum_{x' \in X} P_{Y|X}(y|x')Q(x')} \right)$

where $C$ is the capacity of the channel. Notice that this upper bound to the capacity is independent of the maximizing distribution.

Problem 3.

(a) Show that $I(U; V) \geq I(U; V|T)$ if $T$, $U$, $V$ form a Markov chain, i.e., conditional on $U$, the random variables $T$ and $V$ are independent.

Fix a conditional probability distribution $p(y|x)$, and suppose $p_1(x)$ and $p_2(x)$ are two probability distributions on $X$.

For $k \in \{1, 2\}$, let $I_k$ denote the mutual information between $X$ and $Y$ when the distribution of $X$ is $p_k(x)$.

For $0 \leq \lambda \leq 1$, let $W$ be a random variable, taking values in $\{1, 2\}$, with

$$\Pr(W = 1) = \lambda, \quad \Pr(W = 2) = 1 - \lambda.$$

Define

$$p_{W,X,Y}(w, x, y) = \begin{cases} \lambda p_1(x)p(y|x) & \text{if } w = 1 \\ (1 - \lambda)p_2(x)p(y|x) & \text{if } w = 2. \end{cases}$$

(b) Express $I(X; Y|W)$ in terms of $I_1$, $I_2$ and $\lambda$.

(c) Express $p(x)$ in terms of $p_1(x)$, $p_2(x)$ and $\lambda$. 

(d) Using (a), (b) and (c) show that, for every fixed conditional distribution \( p_{Y|X} \), the mutual information \( I(X;Y) \) is a concave \( \cap \) function of \( p_X \).

**Problem 4.** Suppose \( Z \) is uniformly distributed on \([-1,1]\), and \( X \) is a random variable, independent of \( Z \), constrained to take values in \([-1,1]\). What distribution for \( X \) maximizes the entropy of \( X+Z \)? What distribution of \( X \) maximizes the entropy of \( XZ \)?

**Problem 5.** Let \( P_1 \) and \( P_2 \) be two channels of input alphabet \( \mathcal{X}_1 \) and \( \mathcal{X}_2 \) and of output alphabet \( \mathcal{Y}_1 \) and \( \mathcal{Y}_2 \) respectively. Consider a communication scheme where the transmitter chooses the channel (\( P_1 \) or \( P_2 \)) to be used and where the receiver knows which channel were used. This scheme can be formalized by the channel \( P \) of input alphabet \( \mathcal{X} = (\mathcal{X}_1 \times \{1\}) \cup (\mathcal{X}_2 \times \{2\}) \) and of output alphabet \( \mathcal{Y} = (\mathcal{Y}_1 \times \{1\}) \cup (\mathcal{Y}_2 \times \{2\}) \), which is defined as follows:

\[
P(y,k'|x,k) = \begin{cases} 
P_{k}(y|x) & \text{if } k' = k, \\ 0 & \text{otherwise.} \end{cases}
\]

Let \( X = (X_k, K) \) be a random variable in \( \mathcal{X} \) which will be the input distribution to the channel \( P \), and let \( Y = (Y_k, K) \in \mathcal{Y} \) be the output distribution. Define \( X_1 \) as being the random variable in \( \mathcal{X}_1 \) obtained by conditioning \( X_k \) on \( K = 1 \). Similarly define \( X_2, Y_1 \) and \( Y_2 \). Let \( \alpha \) be the probability that \( K = 1 \).

(a) Show that \( I(X;Y) = h_2(\alpha) + \alpha I(X_1;Y_1) + (1 - \alpha)I(X_2;Y_2). \)

(b) What is the input distribution \( X \) that achieves the capacity of \( P \)?

(c) Show that the capacity \( C \) of \( P \) satisfies \( 2^C = 2^{C_1} + 2^{C_2} \), where \( C_1 \) and \( C_2 \) are the capacities of \( P_1 \) and \( P_2 \) respectively.

**Problem 6.** Suppose \( X \) and \( Y \) are independent geometric random variables. That is, \( p_X(k) = (1 - p)^{k-1}p \) and \( p_Y(k) = (1 - q)^{k-1}q, \ \forall k \in \{1,2,\ldots\}. \)

(a) Find \( H(X,Y) \).

(b) Find \( H(2X+Y, X-2Y) \)

Now consider two independent exponential random variables \( X \) and \( Y \). That is, \( p_X(t) = \lambda_X e^{-\lambda_X t} \) and \( p_Y(t) = \lambda_Y e^{-\lambda_Y t}, \ \forall t \in [0,\infty) \).

(c) Find \( h(X,Y) \).

(d) Find \( h(2X+Y, X-2Y) \)