Problem 1.

(a) We have \( H(f(U)) \leq H(f(U), U) = H(U) + H(f(U)|U) = H(U) + 0 = H(U) \).

(b) Notice that \( U \not\rightarrow V \not\rightarrow f(V) \) is a Markov chain. The data processing inequality implies that \( H(U) - H(U|f(V)) = I(U; f(V)) \leq I(U; V) = H(U) - H(U|V) \). Therefore, \( H(U|V) \leq H(U|f(V)) \).

Problem 2.

(a) We have:
\[
H(U|\hat{U}) \leq H(U,W|\hat{U}) = H(W|\hat{U}) + H(U|\hat{U},W) \leq H(W) + H(U|\hat{U},W)
\]
\[
= H(W) + H(U|\hat{U},W = 0) \cdot \mathbb{P}[W = 0] + H(U|\hat{U},W = 1) \cdot \mathbb{P}[W = 1]
\]
\[
\leq h_2(p_e) + \log(|\mathcal{U}| - 1) \cdot p_e = h_2(p_e) + p_e \log(|\mathcal{U}| - 1),
\]
where \((*)\) follows from the following facts:

- \( H(W) = h_2(p_e) \).
- \( H(U|\hat{U},W = 0) = 0 \): conditioned on \( W = 0 \), we know that \( U = \hat{U} \) and so the conditional entropy \( H(U|\hat{U},W = 0) \) is equal to 0.
- \( H(U|\hat{U},W = 1) \leq \log(|\mathcal{U}| - 1) \): conditioned on \( W = 1 \), we know that \( U \neq \hat{U} \) and so there are at most \( |\mathcal{U}| - 1 \) values for \( U \). Therefore, the conditional entropy \( H(U|\hat{U},W = 1) \) is at most \( \log(|\mathcal{U}| - 1) \).

(b) Let \( \hat{U} = f(V) \). We have \( H(U|\hat{U}) \leq h_2(p_e) + p_e \log(|\mathcal{U}| - 1) \) from (a). On the other hand, from Problem 1(b) we have \( H(U|V) \leq H(U|f(V)) = H(U|\hat{U}) \). We conclude that \( H(U|V) \leq h_2(p_e) + p_e \log(|\mathcal{U}| - 1) \).

Problem 3.

(a) Since
\[
P(U = u, Z = z) = \begin{cases} p(u) & \text{if } z = 1, \\ q(u) & \text{if } z = 2, \end{cases}
\]
one can immediately see that the distribution of \( U \) is \( r(u) = \theta p(u) + (1 - \theta)q(u) \).

(b) \( H(U) = h(r) \), and
\[
H(U|Z) = \sum_z P(Z = z)H(U|Z = z) = \theta h(p) + (1 - \theta)h(q).
\]
The last equality follows since given \( z = 1 \) (resp. \( z = 2 \)) \( U \) has distribution \( p \) (resp. \( q \)).
Since \( H(U) \geq H(U|Z) \), we have proved that \( h(r) \geq \theta h(p) + (1 - \theta)h(q) \).
Problem 4.

(a) We have:

\[ S = \sum_{u \in \mathcal{U}} \max\{P_1(u), P_2(u)\} \leq \sum_{u \in \mathcal{U}} (P_1(u) + P_2(u)) \]

\[ = \sum_{u \in \mathcal{U}} P_1(u) + \sum_{u \in \mathcal{U}} P_2(u) = 1 + 1 = 2, \]

It is easy to see from (a) that \( S = 2 \) if and only if \( \max\{P_1(u), P_2(u)\} = P_1(u) + P_2(u) \) for all \( u \in \mathcal{U} \), which is equivalent to say that there is no \( u \in \mathcal{U} \) for which we have \( P_1(u) > 0 \) and \( P_2(u) > 0 \). In other words, \( S = 2 \) if and only if

\[ \{ u \in \mathcal{U} : P_1(u) > 0 \} \cap \{ u \in \mathcal{U} : P_2(u) > 0 \} = \emptyset. \]

(b) Let \( l_i = \lceil \log_2 \frac{S}{\max\{P_1(a_i), P_2(a_i)\}} \rceil \), and let us compute the Kraft sum:

\[ \sum_{i=1}^{M} 2^{-l_i} \leq \sum_{i=1}^{M} 2^{-\log_2 \frac{S}{\max\{P_1(a_i), P_2(a_i)\}}} = \sum_{i=1}^{M} \max\{P_1(a_i), P_2(a_i)\} \frac{S}{S} = 1. \]

Since the Kraft sum is at most 1, there exists a prefix-free code where the length of the codeword associated to \( a_i \) is \( l_i \).

(c) Since the code constructed in (b) is prefix free, it must be the case that \( I \geq H(U) \). In order to prove the upper bounds, let \( P^* \) be the true distribution (which is either \( P_1 \) or \( P_2 \)). It is easy to see that \( P^*(a_i) \leq \max\{P_1(a_i), P_2(a_i)\} \) for all \( 1 \leq i \leq M \). We have:

\[ l = \sum_{i=1}^{M} P^*(a_i) \cdot l_i = \sum_{i=1}^{M} P^*(a_i) \cdot \lceil \log_2 \frac{S}{\max\{P_1(a_i), P_2(a_i)\}} \rceil \]

\[ < \sum_{i=1}^{M} P^*(a_i) \cdot \left( 1 + \log_2 \frac{S}{\max\{P_1(a_i), P_2(a_i)\}} \right) \]

\[ = \sum_{i=1}^{M} P^*(a_i) \cdot \left( 1 + \log S + \log_2 \frac{1}{\max\{P_1(a_i), P_2(a_i)\}} \right) \]

\[ = 1 + \log S + \sum_{i=1}^{M} P^*(a_i) \cdot \log_2 \frac{1}{\max\{P_1(a_i), P_2(a_i)\}} \]

\[ \leq 1 + \log S + \sum_{i=1}^{M} P^*(a_i) \cdot \log_2 \frac{1}{P^*(a_i)} = H(U) + \log S + 1 \leq H(U) + 2, \]

where the inequality (*) uses the fact that \( P^*(a_i) \leq \max\{P_1(a_i), P_2(a_i)\} \) for all \( 1 \leq i \leq M \).

(d) Now let \( l_i = \lceil \log_2 \frac{S}{\max\{P_1(a_i), \ldots, P_k(a_i)\}} \rceil \), and let us compute the Kraft sum:

\[ \sum_{i=1}^{M} 2^{-l_i} \leq \sum_{i=1}^{M} 2^{-\log_2 \frac{S}{\max\{P_1(a_i), \ldots, P_k(a_i)\}}} = \sum_{i=1}^{M} \frac{\max\{P_1(a_i), \ldots, P_k(a_i)\}}{S} = 1. \]
Since the Kraft sum is at most 1, there exists a prefix-free code where the length of the codeword associated to $a_i$ is $l_i$. Since the code is prefix free, it must be the case that $\bar{l} \geq H(U)$. In order to prove the upper bounds, let $P^*$ be the true distribution (which is either $P_1$ or . . . or $P_k$). It is easy to see that $P^*(a_i) \leq \max\{P_1(a_i), \ldots, P_k(a_i)\}$ for all $1 \leq i \leq M$. We have:

$$\bar{l} = \sum_{i=1}^{M} P^*(a_i) \cdot l_i = \sum_{i=1}^{M} P^*(a_i) \left[ \log_2 \frac{S}{\max\{P_1(a_i), \ldots, P_k(a_i)\}} \right]$$

$$\leq \sum_{i=1}^{M} P^*(a_i) \left( 1 + \log_2 \frac{S}{\max\{P_1(a_i), \ldots, P_k(a_i)\}} \right)$$

$$= \sum_{i=1}^{M} P^*(a_i) \left( 1 + \log_2 S + \log_2 \frac{1}{\max\{P_1(a_i), \ldots, P_k(a_i)\}} \right)$$

$$= 1 + \log_2 S + \sum_{i=1}^{M} P^*(a_i) \cdot \log_2 \frac{1}{P^*(a_i)} \leq H(U) + \log_2 S + 1,$$

where the inequality ($\ast$) uses the fact that $P^*(a_i) \leq \max\{P_1(a_i), \ldots, P_k(a_i)\}$ for all $1 \leq i \leq M$. Now notice that $\max\{P_1(a_i), \ldots, P_k(a_i)\} \leq \sum_{j=1}^{k} P_j(a_i)$ for all $1 \leq i \leq M$. Therefore, we have

$$S = \sum_{i=1}^{M} \max\{P_1(a_i), \ldots, P_k(a_i)\} \leq \sum_{i=1}^{M} \sum_{j=1}^{k} P_j(a_i) = \sum_{j=1}^{k} \sum_{i=1}^{M} P_j(a_i) = k \cdot \sum_{j=1}^{k} 1 = k.$$

We conclude that $H(U) \leq \bar{l} \leq H(U) + \log S + 1 \leq H(U) + \log k + 1$.

**Problem 5.**

(a) We prove the identity by induction on $n \geq 1$. For $n = 1$, the identity is trivial. Let $n > 1$ and suppose that the identity is true up to $n - 1$. We have:

$$I(Y_1^n; X_n) = I(Y_1^{n-1}, Y_n; X_n) = I(Y_1^{n-2}, X_n) + I(X_n; Y_n; Y_1^{n-2})$$

$$\overset{(\ast)}{=} I(Y_1^{n-2}; X_n) + I(X_n; Y_n; Y_1^{n-2}) = \sum_{i=1}^{n-1} I(X_n; Y_i|Y_1^{i-1}).$$

The identity ($\ast$) is by the chain rule for mutual information, and the identity (**) is by the induction hypothesis.

(b) For every $0 \leq i \leq n$, define $a_i = I(X_i^n; Y_i^i)$, and for every $1 \leq i \leq n$, define $b_i = I(X_i^{i+1}; Y_i^{i-1})$. It is easy to see that $a_0 = a_n = 0$. We have:

$$\sum_{i=1}^{n} I(X_i^{i+1}; Y_i|Y_1^{i-1}) = \sum_{i=1}^{n} \left( I(X_i^{i+1}; Y_i) - I(X_i^{i+1}; Y_1^{i-1}) \right) = \left( \sum_{i=1}^{n} a_i \right) - \left( \sum_{i=1}^{n} b_i \right)$$

$$\overset{(\ast)}{=} \left( \sum_{i=0}^{n} a_i \right) - \left( \sum_{i=1}^{n} b_i \right) = \left( \sum_{i=1}^{n} a_{i-1} \right) - \left( \sum_{i=1}^{n} b_i \right) = \sum_{i=1}^{n} (a_{i-1} - b_i)$$

$$= \sum_{i=1}^{n} \left( I(X_i^n; Y_1^{i-1}) - I(X_i^{i+1}; Y_1^{i-1}) \right) \overset{(**)}{=} \sum_{i=1}^{n} I(Y_1^{i-1}; X_i|X_i^{i+1}).$$
Problem 6.

(a) We can write the following chain of inequalities:

\[
Q^n(x) = \prod_{i=1}^{n} Q(x_i) = \prod_{a \in \mathcal{X}} Q(a)^{N(a|x)} = \prod_{a \in \mathcal{X}} Q(a)^{nP_x(a)} = \prod_{a \in \mathcal{X}} 2^{nP_x(a) \log Q(a)}
\]

where 1 follows because the sequence is i.i.d., grouping symbols gives 2, and 3 is the definition of type.

(b) Upper bound: We know that

\[
1 \geq \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = 1.
\]

Consider one term and set \( p = k/n \). Then,

\[
1 \geq \binom{n}{k} \left( \frac{k}{n} \right)^k \left( 1 - \frac{k}{n} \right)^{n-k} = \binom{n}{k} 2^{n\left( \frac{k}{n} \log \frac{k}{n} + \frac{n-k}{n} \log \frac{n-k}{n} \right)} = \binom{n}{k} 2^{-nh_2(k)}
\]

Lower bound: Define \( S_j = \binom{n}{j} p^j (1-p)^{n-j} \). We can compute

\[
\frac{S_{j+1}}{S_j} = \frac{n-j}{j+1} \frac{p}{1-p}.
\]

One can see that this ratio is a decreasing function in \( j \). It equals 1, if \( j = np + p - 1 \), so \( \frac{S_{j+1}}{S_j} < 1 \) for \( j = \lceil np + p \rceil \) and \( \frac{S_{j+1}}{S_j} \geq 1 \) for any smaller \( j \). Hence, \( S_j \) takes its maximum value at \( j = \lceil np + p \rceil \), which equals \( kn \) in our case. From this we have that

\[
1 = \sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j} \leq (n+1) \max_j \binom{n}{j} p^j (1-p)^j \leq (n+1) \binom{n}{k} \left( \frac{k}{n} \right)^k \left( 1 - \frac{k}{n} \right)^{n-k} = (n+1) \binom{n}{k} 2^{-nh_2(k)}.
\]

The last equality comes from the derivation we had when proving the upper bound.

(c) Since for every \( x \in T(P) \), \( Q^n(x) = 2^{-n(H(P) + D(P||Q))} \) (by part (a)) and \( \frac{1}{n+1} 2^{nH(P)} \leq |T(P)| \leq 2^{nH(P)} \) (by part (b)), we have

\[
\frac{1}{n+1} 2^{-nD(P||Q)} \leq Q^n(T(P)) \leq 2^{-nD(P||Q)}
\]