Problem 1.

(a) A source has an alphabet of 4 letters, \(a_1, a_2, a_3, a_4\), and we have the condition \(P(a_1) > P(a_2) = P(a_3) = P(a_4)\). Find the smallest number \(q\) such that \(P(a_1) > q\) implies that \(n_1 = 1\) where \(n_1\) throughout this problem is the length of the codeword for \(a_1\) in a Huffman code.

(b) Show by example that if \(P(a_1) = q\) (your answer in part (a)), then a Huffman code exists with \(n_1 > 1\).

(c) Now assume the more general condition, \(P(a_1) > P(a_2) \geq P(a_3) \geq P(a_4)\). Does \(P(a_1) > q\) still imply that \(n_1 = 1\)? Why or why not?

(d) Now assume that the source has an arbitrary number \(K\) of letters with \(P(a_1) > P(a_2) \geq \cdots \geq P(a_K)\). Does \(P(a_1) > q\) now imply \(n_1 = 1\)?

(e) Assume \(P(a_1) \geq P(a_2) \geq \cdots \geq P(a_K)\). Find the largest number \(q'\) such that \(P(a_1) < q'\) implies that \(n_1 > 1\).

Problem 2. Suppose \(X, Y\) and \(Z\) are random variables.

(a) Show that \(H(X) + H(Y) + H(Z) \geq \frac{1}{2}[H(XY) + H(YZ) + H(ZX)]\).

(b) Show that \(H(XY) + H(YZ) \geq H(XYZ) + H(Y)\).

(c) Show that
\[
\]

(d) Show that \(H(XY) + H(YZ) + H(ZX) \geq 2H(XYZ)\).

(e) Suppose \(n\) points in three dimensions are arranged so that their their projections to the \(xy\), \(yz\) and \(zx\) planes give \(n_{xy}\), \(n_{yz}\) and \(n_{zx}\) points. Clearly \(n_{xy} \leq n\), \(n_{yz} \leq n\), \(n_{zx} \leq n\). Use part (d) show that
\[
n_{xy}n_{yz}n_{zx} \geq n^2.
\]

Problem 3. Let \(X\) be a random variable taking values in \(M\) points \(a_1, \ldots, a_M\), and let \(P_X(a_M) = \alpha\). Show that
\[
H(X) = \alpha \log \frac{1}{\alpha} + (1 - \alpha) \log \frac{1}{1 - \alpha} + (1 - \alpha)H(Y)
\]
where \(Y\) is a random variable taking values in \(M - 1\) points \(a_1, \ldots, a_{M-1}\) with probabilities \(P_Y(a_j) = P_X(a_j)/(1 - \alpha); 1 \leq j \leq M - 1\). Show that
\[
H(X) \leq \alpha \log \frac{1}{\alpha} + (1 - \alpha) \log \frac{1}{1 - \alpha} + (1 - \alpha) \log(M - 1)
\]
and determine the condition for equality.
Problem 4. Let $X, Y, Z$ be discrete random variables. Prove the validity of the following inequalities and find the conditions for equality:

(a) $I(X; Y) \geq I(X; Z)$.

(b) $H(Y) \geq H(Z)$.

(c) $H(X, Y, Z) - H(X, Y) \leq H(X, Z) - H(X)$.

(d) $I(X; Z) \geq I(Z; Y) - I(Z; Y) + I(X; Z)$.

Problem 5. For a stationary process $X_1, X_2, \ldots$, show that

(a) $\frac{1}{n} H(X_1, \ldots, X_n) \geq H(X_n | X_{n-1}, \ldots, X_1)$.

(b) $\frac{1}{n} H(X_1, \ldots, X_n) \leq \frac{1}{n-1} H(X_1, \ldots, X_{n-1})$.

Problem 6. Let $\{X_i\}_{i=-\infty}^{\infty}$ be a stationary stochastic process. Prove that

$$H(X_0 | X_{-1}, \ldots, X_{-n}) = H(X_0 | X_1, \ldots, X_n).$$

That is: the conditional entropy of the present given the past is equal to the conditional entropy of the present given the future.

Problem 7. Let $X \perp Y \perp (Z, W)$ form a Markov chain. Show that

$$I(X; Z) + I(X; W) \leq I(X; Y) + I(Z; W).$$