ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

Handout 6 Information Theory and Coding
Homework 3 Oct. 04, 2022

PROBLEM 1.

(a) A source has an alphabet of 4 letters, a_1, a_2, a_3, a_4, and we have the condition $P(a_1) > P(a_2) = P(a_3) = P(a_4)$. Find the smallest number q such that $P(a_1) > q$ implies that $n_1 = 1$ where n_1 throughout this problem is the length of the codeword for a_1 in a Huffman code.

(b) Show by example that if $P(a_1) = q$ (your answer in part (a)), then a Huffman code exists with $n_1 > 1$.

(c) Now assume the more general condition, $P(a_1) > P(a_2) \geq P(a_3) \geq P(a_4)$. Does $P(a_1) > q$ still imply that $n_1 = 1$? Why or why not?

(d) Now assume that the source has an arbitrary number K of letters with $P(a_1) > P(a_2) \geq \cdots \geq P(a_K)$. Does $P(a_1) > q$ now imply $n_1 = 1$?

(e) Assume $P(a_1) \geq P(a_2) \geq \cdots \geq P(a_K)$. Find the largest number q' such that $P(a_1) < q'$ implies that $n_1 > 1$.

PROBLEM 2. Suppose X,Y and Z are random variables.

(a) Show that $H(X) + H(Y) + H(Z) \geq \frac{1}{2}\big[H(XY) + H(YZ) + H(ZX)\big]$.

(b) Show that $H(XY) + H(YZ) \geq H(XYZ) + H(Y)$.

(c) Show that

$$2\big[H(XY) + H(YZ) + H(ZX)\big] \geq 3H(XYZ) + H(X) + H(Y) + H(Z).$$

(d) Show that $H(XY) + H(YZ) + H(ZX) \geq 2H(XYZ)$.

(e) Suppose n points in three dimensions are arranged so that their their projections to the xy, yz and zx planes give n_{xy}, n_{yz} and n_{zx} points. Clearly $n_{xy} \leq n$, $n_{yz} \leq n$, $n_{zx} \leq n$. Use part (d) show that

$$n_{xy} n_{yz} n_{zx} \geq n^2.$$

PROBLEM 3. Let X be a random variable taking values in M points $a_1, \ldots, a_M$, and let $P_X(a_M) = \alpha$. Show that

$$H(X) = \alpha \log \frac{1}{\alpha} + (1-\alpha) \log \frac{1}{1-\alpha} + (1-\alpha) H(Y)$$

where Y is a random variable taking values in $M-1$ points $a_1, \ldots, a_{M-1}$ with probabilities $P_Y(a_j) = P_X(a_j)/(1-\alpha)$; $1 \leq j \leq M-1$. Show that

$$H(X) \leq \alpha \log \frac{1}{\alpha} + (1-\alpha) \log \frac{1}{1-\alpha} + (1-\alpha) \log(M-1)$$

and determine the condition for equality.

PROBLEM 4. Let X, Y, Z be discrete random variables. Prove the validity of the following inequalities and find the conditions for equality:

(a) $I(X, Y; Z) \geq I(X; Z)$.

(b) $H(X, Y|Z) \geq H(X|Z)$.

(c) $H(X, Y, Z) - H(X, Y) \leq H(X, Z) - H(X)$.

(d) $I(X; Z|Y) \geq I(Z; Y|X) - I(Z; Y) + I(X; Z)$.

PROBLEM 5. For a stationary process $X_1, X_2, \ldots$, show that

(a) $\frac{1}{n} H(X_1, \ldots, X_n) \geq H(X_n | X_{n-1}, \ldots, X_1)$.

(b) $\frac{1}{n} H(X_1, \ldots, X_n) \leq \frac{1}{n-1} H(X_1, \ldots, X_{n-1})$.

PROBLEM 6. Let $\{X_i\}_{i=-\infty}^{\infty}$ be a stationary stochastic process. Prove that

$$H(X_0 | X_{-1}, \ldots, X_{-n}) = H(X_0 | X_1, \ldots, X_n).$$

That is: the conditional entropy of the present given the past is equal to the conditional entropy of the present given the future.

PROBLEM 7. Let $X \multimap Y \multimap (Z, W)$ form a Markov chain. Show that

$$I(X; Z) + I(X; W) \leq I(X; Y) + I(Z; W)$$