

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 4
Homework 2

Information Theory and Coding
Sep. 27, 2022

PROBLEM 1. A source has an alphabet of 4 letters. The probabilities of the letters and two possible sets of binary code words for the source are given below:

Letter	Prob.	Code I	Code II
a_1	0.4	1	1
a_2	0.3	01	10
a_3	0.2	001	100
a_4	0.1	000	1000

For *each* code, answer the following questions (no proofs or numerical answers are required).

- (a) Is the code prefix-free?
N.B. A “prefix-free” code is also known as an *instantaneous* code.
- (b) Is the code uniquely decodable?
- (c) Give an heuristic description of the purpose of the first letter in the code words of code II.

PROBLEM 2. Recall that for a code $\mathcal{C} : \mathcal{U} \rightarrow \{0,1\}^*$, we define $\mathcal{C}^n : \mathcal{U}^n \rightarrow \{0,1\}^*$ as $\mathcal{C}^n(u_1 \dots u_n) = \mathcal{C}(u_1) \dots \mathcal{C}(u_n)$.

- (a) (4 pts) Show that if \mathcal{C} is uniquely decodable, then for all $n \geq 1$, \mathcal{C}^n is injective.
- (b) (4 pts) Suppose \mathcal{C} is not uniquely decodable. Show that there are u^n and v^m such that $u_1 \neq v_1$ and $\mathcal{C}^n(u^n) = \mathcal{C}^m(v^m)$.
- (c) (4 pts) Suppose \mathcal{C} is not uniquely decodable. Show that there is a k such that \mathcal{C}^k is not injective. [Hint: try $k = n + m$.]

PROBLEM 3. Let $\bar{M} = \sum_i p_i l_i^{2015}$ be the 2015th moment (i.e., the expected value of the 2015th power) of the code word lengths l_i associated with an encoding of a random variable X with distribution p . Let $\bar{M}_1 = \min \bar{M}$ over all prefix-free (instantaneous) codes for X ; and let $\bar{M}_2 = \min \bar{M}$ over all uniquely decodable codes for X . What relationship exists between \bar{M}_1 and \bar{M}_2 ?

PROBLEM 4. Consider the following method for constructing binary code words for a random variable U which takes values $\{a_1, \dots, a_m\}$ with probabilities $P(a_1), \dots, P(a_m)$. Assume that $P(a_1) \geq P(a_2) \geq \dots \geq P(a_m)$. Define

$$Q_1 = 0 \quad \text{and} \quad Q_i = \sum_{k=1}^{i-1} P(a_k) \quad \text{for } i = 2, 3, \dots$$

The code word assigned to the letter a_i is formed by finding the binary expansion of $Q_i < 1$ (i.e., $1/2 = .100\dots$, $1/4 = .0100\dots$, $5/8 = .1010\dots$) and letting the codeword be the first l_i bits of this expansion where $l_i = \lceil -\log_2 P(a_i) \rceil$.

- (a) Construct binary code words for the probability distribution $\{1/4, 1/4, 1/8, 1/8, 1/16, 1/16, 1/16, 1/16\}$.
- (b) Prove that the method described above yields a prefix-free (instantaneous) code and the average codeword length \bar{L} satisfies

$$H(X) \leq \bar{L} < H(X) + 1.$$

PROBLEM 5. A random variable takes values on an alphabet of K letters, and each letter has the same probability. These letters are encoded into binary words using the Huffman procedure so as to minimize the average code word length. Let j and x be chosen such that $K = x2^j$, where j is an integer and $1 \leq x < 2$.

- (a) Do any code words have lengths not equal to j or $j + 1$? Why?
- (b) In terms of j and x , how many code words have length j ?
- (c) What is the average code word length?

PROBLEM 6. Consider two discrete memoryless sources. Source 1 has an alphabet of 6 symbols with the probabilities, 0.3, 0.2, 0.15, 0.15, 0.1, 0.1. Source 2 has an alphabet of 7 letters with probabilities 0.3, 0.25, 0.15, 0.1, 0.1, 0.05, 0.05. Construct a binary and a ternary Huffman code for each source. Find the average number of code letters per source symbol in each case.

Hint: A ternary code is a mapping of source symbols to $\{0, 1, 2\}^*$. Observe that a ternary tree has an odd number of leaves. A fictitious symbol with probability 0 might therefore be needed for the code construction.