

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 2
Homework 1

Information Theory and Coding
Sep. 20, 2022

PROBLEM 1. Three events E_1 , E_2 and E_3 , defined on the same probability space, have probabilities $P(E_1) = P(E_2) = P(E_3) = 1/4$. Let E_0 be the event that one or more of the events E_1 , E_2 , E_3 occurs.

(a) Find $P(E_0)$ when:

- (1) The events E_1 , E_2 and E_3 are disjoint.
- (2) The events E_1 , E_2 and E_3 are independent.
- (3) The events E_1 , E_2 and E_3 are in fact three names for the same event.

(b) Find the maximum value $P(E_0)$ can take when:

- (1) Nothing is known about the independence or disjointness of E_1 , E_2 , E_3 .
- (2) It is known that E_1 , E_2 and E_3 are *pairwise independent*, i.e., that the probability of realizing both E_i and E_j is $P(E_i)P(E_j)$, $1 \leq i \neq j \leq 3$, but nothing is known about the probability of realizing all three events together.

(c) Suppose now that events E_1 , E_2 and E_3 all have probability p , that they are pairwise independent, and that E_0 has probability 1. Show that p has to be at least $1/2$.

PROBLEM 2. A child is playing a game and tosses a fair die until the first 6 comes. Here, the number of tosses is a random variable denoted by N_1 . N_1 takes values in $\{1, 2, \dots\}$

(a) Find $P(N_1 = k)$, $k \in \{1, 2, \dots\}$

(b) Find $E[N_1]$. (*Hint*: $\sum_{k=1}^{\infty} x^{k-1}k = 1/(1-x)^2$)

(c) The child tries to make the game a little bit longer. Now, he stops the game when he gets the m^{th} 6. For example, when $m = 2$, he stops when the observed sequence is 1, 4, 2, 3, 6, 2, 3, 4, 6. Denote the new random variable by \tilde{N} , where \tilde{N} takes values in $\{m, m+1, \dots\}$. Repeat (a) and (b) for this case.

(d) This child has a older brother and he has a loaded dice with identical appearance and $P(\text{Top face shows } 6) = 1/6^5$. He takes the fair dice from his little brother and puts both die in a bag. The child then chooses a die at random. Suppose that he observes the first 6 at k^{th} outcome. Based on this observation, what is the posterior probability that the die is fair? For which range of k is $P(\text{Fair} \mid N_1 = k) < P(\text{Loaded} \mid N_1 = k)$?

PROBLEM 3. Suppose the random variables A , B , C , D form a Markov chain: $A \leftrightarrow B \leftrightarrow C \leftrightarrow D$.

(a) Is $A \leftrightarrow B \leftrightarrow C$?

(b) Is $B \leftrightarrow C \leftrightarrow D$?

(c) Is $A \perp (B, C) \perp D$?

PROBLEM 4. Suppose the random variables A, B, C, D satisfy $A \perp B \perp C$, and $B \perp C \perp D$. Does it follow from these that $A \perp B \perp C \perp D$?

PROBLEM 5. Let X and Y be two random variables.

- (a) Prove that the expectation of the sum of X and Y , $E[X + Y]$, is equal to the sum of the expectations, $E[X] + E[Y]$.
- (b) Prove that if X and Y are independent, then X and Y are also uncorrelated (by definition X and Y are uncorrelated if $E[XY] = E[X]E[Y]$). Find an example in which X and Y are dependent yet uncorrelated.
- (c) Prove that if X and Y are independent, then the variance of the sum $X + Y$ is equal to the sum of variances. Is this relationship valid if X and Y are uncorrelated but not independent?

PROBLEM 6. After summer, the winter tyres of a car (with four wheels) are to be put back. However, the owner has forgotten which tyre goes to which wheel, and the tyres are installed ‘randomly’, each of the $4! = 24$ permutations being equally likely.

- (a) What is the probability that tyre 1 is installed in its original position?
- (b) What is the probability that all the tyres are installed in their original positions?
- (c) What is the expected number of tyres that are installed in their original positions?
- (d) Redo the above for a vehicle with n wheels.
- (e) (Harder.) What is the probability that none of the wheels are installed in their original positions.

PROBLEM 7. We construct an ‘inventory’ by drawing n independent samples from a distribution p . Let X_1, \dots, X_n be the random variables that represent the drawings.

Suppose X is drawn from distribution p , independent of X_1, \dots, X_n .

- (a) What is the probability that X does not appear in the inventory?
- (b) Redo (a) for the special case when p is the uniform distribution over n items.
- (c) What happens to the probability in (b) when n gets large?