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**Handout 19** Principles of Digital Communications
Solutions to Problem Set 8 Apr. 21, 2021

SOLUTION 1. First we compute $T_s$, which is the duration of one bit:

$$T_s = \frac{1}{1 \text{ Mbps}} = 10^{-6} \text{ s}.$$

Now, we can calculate the energy of the signal (i.e. the energy per bit), which is the same for every $j$:

$$\mathcal{E}_b = b^2 T_s.$$

The bit error probability is given by $Q\left(\frac{\sqrt{\mathcal{E}_b}}{\sigma}\right)$. In our case $\sigma = \sqrt{N_0/2} = 10^{-1}$, thus we need to solve

$$10^{-5} = Q\left(\frac{10^{-3} \times b}{10^{-1}}\right) = Q\left(10^{-2} \times b\right),$$

hence $b = Q^{-1}(10^{-5}) \times 10^2 \approx 426.5$.

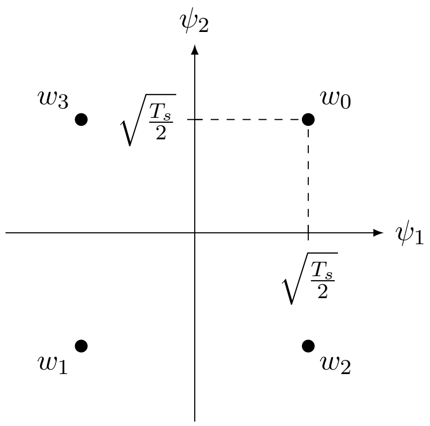
SOLUTION 2.

(a) There are various possibilities to choose an orthogonal basis. One is $\phi_1(t) = \frac{w_0(t)}{\|w_0\|} = \sqrt{\frac{1}{T_s}} w_0(t)$ and $\phi_2(t) = \frac{w_2(t)}{\|w_2\|} = \sqrt{\frac{1}{T_s}} w_2(t)$. Another choice, that we prefer and will be our choice in this solution is

$$\psi_1(t) = \sqrt{\frac{2}{T_s}} \mathbb{1}_{[0, \frac{T_s}{2}]}(t)$$
$$\psi_2(t) = \sqrt{\frac{2}{T_s}} \mathbb{1}_{[\frac{T_s}{2}, T_s]}(t).$$

With the latter choice the signal space is

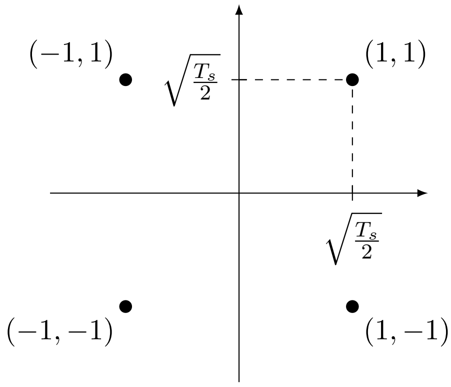
$$w_0 = \sqrt{\frac{T_s}{2}}(1,1)^\mathsf{T} \qquad w_2 = \sqrt{\frac{T_s}{2}}(1,-1)^\mathsf{T}$$
$$w_1 = \sqrt{\frac{T_s}{2}}(-1,-1)^\mathsf{T} \qquad w_3 = \sqrt{\frac{T_s}{2}}(-1,1)^\mathsf{T}$$



(b) $U_0 \in \{\pm 1\}$ and $U_1 \in \{\pm 1\}$ are mapped into

$$U_0\sqrt{\frac{T_s}{2}}\psi_1(t) + U_1\sqrt{\frac{T_s}{2}}\psi_2(t).$$

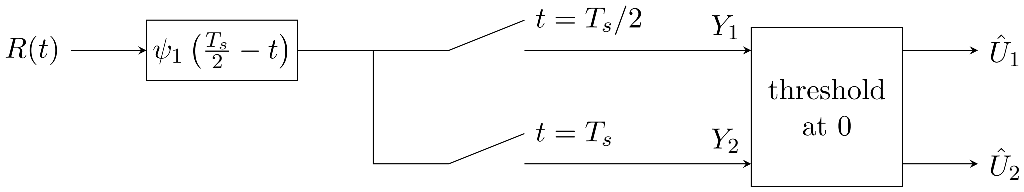
The mapping is shown below:




The mapping is such that neighboring points differ by one bit. This minimizes the bit-error probability since when we make an error chances are that we choose a neighbor of the correct symbol. Notice that we may decode each bit independently. In fact the first bit is decoded to a 1 iff the observation is to the right of the vertical axis and the second bit is 1 iff it is above the horizontal axis. The bit error probability is therefore

$$P_b = Q\left(\frac{\sqrt{T_s/2}}{\sqrt{N_0/2}}\right) = Q\left(\sqrt{\frac{T_s}{N_0}}\right).$$

(c) Notice that $\psi_2(t) = \psi_1(t - \frac{T_s}{2})$. Hence one matched filter is enough. The receiver block diagram is:




(d) $\mathcal{E}_b = \frac{\mathcal{E}_s}{2} = \frac{T_s}{2}$ and the power is $\frac{\mathcal{E}_s}{T_s} = 1$.

Solution 3.

(a) Using the identity $\cos^2(a) = \frac{1}{2}[1 + \cos(2a)]$, the average energy can be computed as

$$\begin{aligned}\int_{-\infty}^{\infty} |w_i(t)|^2\, dt &= \frac{2\mathcal{E}}{T}\int_0^T \cos^2(2\pi(f_c + i\Delta f)t)\, dt \\ &= \frac{2\mathcal{E}}{T}\left[\frac{t}{2} + \frac{\sin(4\pi(f_c + i\Delta f)t)}{8\pi(f_c + i\Delta f)}\right]_0^T \\ &= \mathcal{E}\left[1 + \frac{\sin(4\pi i\Delta f T)}{4\pi(f_c + i\Delta f)}\right] \approx \mathcal{E}. \qquad (*)\end{aligned}$$

The last approximation follows since $f_c \gg \Delta f$ implies the second term in the square brackets is negligible.

(b) Orthogonality requires

$$\mathcal{E}\frac{2}{T}\int_0^T \cos(2\pi(f_c+i\Delta f)t)\cos(2\pi(f_c+j\Delta f)t)\ dt = 0,$$

for every $i \neq j$. Using the trigonometric identity $\cos(\alpha)\cos(\beta) = \frac{1}{2}\cos(\alpha+\beta)+\frac{1}{2}\cos(\alpha-\beta)$, an equivalent condition is

$$\frac{\mathcal{E}}{T}\int_0^T [\cos(2\pi(i-j)\Delta f t) + \cos(2\pi(2f_c+(i+j)\Delta f)t)]\ dt = 0.$$

Integrating we obtain

$$\frac{\mathcal{E}}{T}\left[\frac{\sin(2\pi(i-j)\Delta f T)}{2\pi(i-j)\Delta f} + \frac{\sin(2\pi(2f_c+(i+j)\Delta f)T)}{2\pi(2f_c+(i+j)\Delta f)}\right] = 0.$$

As $f_c T$ is assumed to be an integer, the result can be simplified to

$$\frac{\mathcal{E}}{T}\left[\frac{\sin(2\pi(i-j)\Delta f T)}{2\pi(i-j)\Delta f} + \frac{\sin(2\pi(i+j)\Delta f T)}{2\pi(2f_c+(i+j)\Delta f)}\right] = 0.$$

As $i$ and $j$ are integer, this is satisfied for $i \neq j$ if and only if $2\pi\Delta f T$ is an integer multiple of $\pi$. Hence, we obtain the minimum value of $\Delta f$ if $2\pi\Delta f T = \pi$ which gives $\Delta f = \frac{1}{2T}$. Note that once $\Delta f$ is an integer multiple of $\frac{1}{2T}$ the approximate equality in $(*)$ will be exact.

(c) Proceeding similarly, we will have orthogonality if and only if

$$\frac{\mathcal{E}}{T}\Bigg[\frac{\sin(2\pi(i-j)\Delta f T+\theta_i-\theta_j)-\sin(\theta_i-\theta_j)}{2\pi(i-j)\Delta f} + \frac{\sin(2\pi(i+j)\Delta f T+\theta_i+\theta_j)-\sin(\theta_i+\theta_j)}{2\pi(2f_c+(i+j)\Delta f)}\Bigg] = 0.$$

In this case we see that both parts become zero if and only if $2\pi\Delta f T$ is an even multiple of $\pi$, meaning that the smallest $\Delta f$ is $\Delta f = \frac{1}{T}$ which is twice the minimum frequency separation needed in the previous part. Hence, the cost of phase uncertainty is a bandwidth expansion by a factor of 2.

(d) The condition for essential orthogonality is that

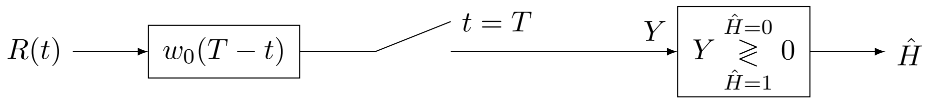
$$\frac{\mathcal{E}}{T}\left[\frac{\sin(2\pi(i-j)\Delta f T+\theta_i-\theta_j)-\sin(\theta_i-\theta_j)}{2\pi(i-j)\Delta f}\right] + \frac{\mathcal{E}}{T}\left[\frac{\sin(2\pi(2f_c(i+j)\Delta f T)+\theta_i+\theta_j)-\sin(\theta_i+\theta_j)}{2\pi(2f_c+(i+j)\Delta f)}\right]$$

is small compared to the signal's energy $\mathcal{E}$. The first term vanishes if $\Delta f = \frac{1}{T}$. The second term is very small compared to $\mathcal{E}$ if $f_c T \gg 1$.

(e) We have $m$ signals separated by $\Delta f$. The approximate bandwidth is $m\Delta f$. This means bandwidth $\frac{2^k}{2T}$ without random phase, and bandwidth $\frac{2^k}{T}$ with random phase. We see that in both cases, $WT$ is proportional to $2^k$, i.e. it grows exponentially with $k$.

Solution 4.

(a) The block diagram is shown below:




(b) Given $A=a$, the distance of signals is $2a\sqrt{\mathcal{E}_b}$, hence

$$P_e(a) = Q\left(a\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right).$$

(c)

$$P_f = \mathbb{E}[P_e(a)] = \int_0^\infty Q\left(a\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right) 2ae^{-a^2}\ da.$$

We integrate by parts, noting that $\int 2ae^{-a^2}\ da = -e^{-a^2}$:

$$P_f = -Q\left(a\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)e^{-a^2}\Bigg|_0^\infty + \int_0^\infty Q'\left(a\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)e^{-a^2}\ da.$$

Taking the derivative of an integral with respect to the lower boundary gives the negative of the value of the integrand evaluated at the lower boundary, i.e.,

$$Q'(x) = -\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}.$$

Thus, for the derivative of $Q\left(a\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)$ with respect to $a$, we can write

$$\frac{d}{da}Q\left(a\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right) = -\frac{1}{\sqrt{2\pi}}e^{-\frac{a^2\mathcal{E}_b}{N_0}}\sqrt{\frac{2\mathcal{E}_b}{N_0}}.$$

Plugging this in, we find

$$P_f = \frac{1}{2} - \int_0^\infty \frac{1}{\sqrt{2\pi}}\sqrt{\frac{2\mathcal{E}_b}{N_0}}e^{-a^2\left(\frac{\mathcal{E}_b}{N_0}+1\right)}\ da,$$

which we now reshape to make it an integral over a Gaussian density, as follows:

$$P_f = \frac{1}{2} - \sqrt{\frac{2\mathcal{E}_b}{N_0}}\frac{1}{\sqrt{2\left(\frac{\mathcal{E}_b}{N_0}+1\right)}}\int_0^\infty \frac{1}{\sqrt{\frac{\pi}{\left(\frac{\mathcal{E}_b}{N_0}+1\right)}}}\exp\left(-\frac{a^2}{2\frac{1}{2\left(\frac{\mathcal{E}_b}{N_0}+1\right)}}\right)\ da.$$

Now, it is clear that the integral evaluates to one half (since the integral is only over half of the real line), and we find

$$P_f = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{\mathcal{E}_b/N_0}{1+\mathcal{E}_b/N_0}} = \frac{1}{2}\left(1-\sqrt{\frac{\mathcal{E}_b/N_0}{1+\mathcal{E}_b/N_0}}\right).$$

(d) Let $\sigma = \frac{1}{\sqrt{2}}$, then

$$m = \mathbb{E}[A] = \int_0^\infty 2a^2 e^{-a^2} \underset{\cdot}{a} = 2\sqrt{\pi}\int_0^\infty a^2 \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{a^2}{2\sigma^2}}\, da = \sqrt{\pi}\sigma^2 = \frac{\sqrt{\pi}}{2}.$$

Thus, using the formula from part (b):

$$P_e(m) = Q\left(m\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right) = Q\left(\sqrt{\frac{\pi}{2}}\sqrt{\frac{\mathcal{E}_b}{N_0}}\right).$$

For the given example we get

$$\frac{\mathcal{E}_b}{N_0} = \frac{2\left(Q^{-1}(10^{-5})\right)^2}{\pi} \approx 10.6 \text{ dB}.$$

For the fading we use the result of part (c) to get

$$\frac{\mathcal{E}_b}{N_0} = \frac{\left(1 - 2\cdot 10^{-5}\right)^2}{1 - \left(1 - 2\times^{-5}\right)^2} \approx 44 \text{ dB}.$$

The difference is quite significant! It is clear that this behaviour is fundamentally different from the non-fading case.

Solution 5.

(a) In this basis the signal representations are $c_1 = (2,0,0,2)^\mathsf{T}$, $c_2 = (0,2,2,0)^\mathsf{T}$, $c_3 = (2,0,2,0)^\mathsf{T}$, $c_4 = (0,2,0,2)^\mathsf{T}$.

(b) The union bound is expressed in terms of the pairwise distances $d_{ij}$ between the signals since

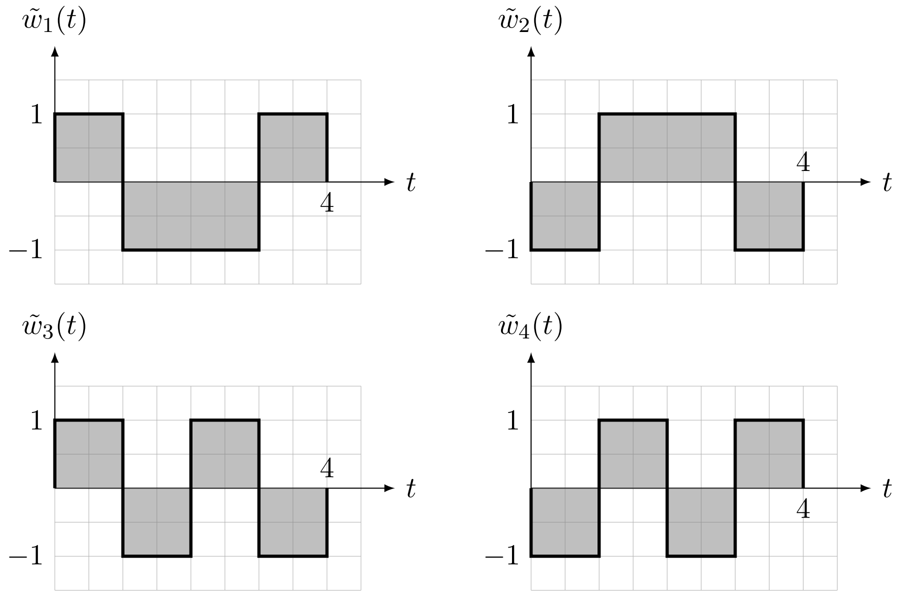
$$P_e(i) \le \sum_{j\neq i} Q\left(\frac{d_{ij}}{2\sigma}\right)$$

From (a) we observe that $d_{12}^2 = d_{34}^2 = 16$ and $d_{13}^2 = d_{14}^2 = d_{23}^2 = d_{24}^2 = 8$, hence

$$P_e(i) \le 2Q\left(\frac{2}{\sqrt{N_0}}\right) + Q\left(\frac{2\sqrt{2}}{\sqrt{N_0}}\right)$$

Since $P_e(i)$ does not depend on $i$, it also bounds the average error probability.

(c) The minimum-energy signal set is obtained by subtracting from $\{w_i(t)\}_{i=1}^4$ the average signal $a(t) = \frac{1}{4}\sum_{i=1}^4 w_i(t) = \mathbb{1}_{[0,4]}(t)$. The resulting signals are shown below.




(d) Note that in the new signal set $\tilde{w}_2(t) = -\tilde{w}_1(t)$ and $\tilde{w}_4(t) = -\tilde{w}_3(t)$. Furthermore the signals $\tilde{w}_1(t)$ and $\tilde{w}_3(t)$ are orthogonal. Thus the new signal space is two-dimensional, and the Gram–Schmidt procedure will produce the orthonormal basis $\tilde{\psi}_1(t) = \frac{\tilde{w}_1(t)}{\|\tilde{w}_1\|} = \frac{1}{2}\tilde{w}_1(t)$ and $\tilde{\psi}_2(t) = \frac{\tilde{w}_3(t)}{\|\tilde{w}_3\|} = \frac{1}{2}\tilde{w}_3(t)$.

(e) In the new basis the signal representations are $\tilde{c}_1 = (2,0)^\mathsf{T}$, $\tilde{c}_2 = (-2,0)^\mathsf{T}$, $\tilde{c}_3 = (0,2)^\mathsf{T}$, $\tilde{c}_4 = (0,-2)^\mathsf{T}$. These codewords correspond to those of the 4-QAM constellation (rotated by 45 degrees). The error probability of this set is

$$P_e = 1 - \left[1 - Q\left(\frac{2}{\sqrt{N_0}}\right)\right]^2 = 2Q\left(\frac{2}{\sqrt{N_0}}\right) - Q\left(\frac{2}{\sqrt{N_0}}\right)^2$$

(f) Since translations of a signal set do not change the probability of error, the error probability of the receiver in (b) is equal to that in (e).

Solution 6.

(a) Clearly,

$$\mathcal{E}_s^C(k) = 2^{2k} - 1.$$

(b)

$$a = Q^{-1}\left(\frac{10^{-5}}{2}\right) \approx 4.42.$$

(From the suggested approximation we get $a \approx 4.80$.)

(c) For comparison, see the following table.

| $k$ | $\mathcal{E}_s^P(k)$ | $\mathcal{E}_s^C(k)$ |
|---|---|---|
| 1 | 19.54 | 3 |
| 2 | 97.68 | 15 |
| 4 | 1660 | 255 |

(d) We see that

$$\frac{\mathcal{E}_s^C(k+1)}{\mathcal{E}_s^C(k)} = \frac{\mathcal{E}_s^P(k+1)}{\mathcal{E}_s^P(k)} = \frac{2^{2(k+1)}-1}{2^{2k}-1},$$

thus

$$\lim_{k\to\infty} \frac{\mathcal{E}_s^C(k+1)}{\mathcal{E}_s^C(k)} = \lim_{k\to\infty} \frac{\mathcal{E}_s^P(k+1)}{\mathcal{E}_s^P(k)} = 4.$$

(e) If we send one bit per symbol, then coding allows us to significantly reduce the required energy per symbol. For every additional bit per symbol we need to multiply $\mathcal{E}_s$ by roughly 4 (exactly 4 asymptotically) with or without coding. So as the number of bits per symbol increases, there is essentially a constant gap (in dB) between the energy per symbol required by (uncoded) PAM and that required by the best possible code.

Notice that to keep the error probability at a constant level, we need to increase $\mathcal{E}_s/\sigma^2$ exponentially with the number $k$ of bits per symbol. In Example 4.3 in the book we increase it linearly with $k$ (hence the error probability goes to 1).