ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 18 | Principles of Digital Communications

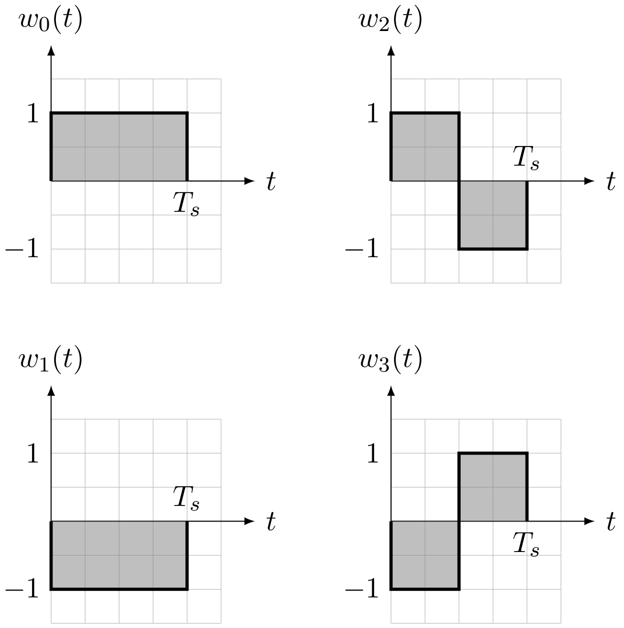
Problem Set 8 | Apr. 21, 2021

PROBLEM 1. A communication system uses bit-by-bit on a pulse train to communicate at 1 Mbps using a rectangular pulse. The transmitted signal is of the form

$$\sum_j B_j \mathbb{1}_{[0,T_s)}(t - jT_s),$$

where $B_j \in \{\pm b\}$. Determine the value of b needed to achieve bit-error probability $P_b = 10^{-5}$ knowing that the channel corrupts the transmitted signal with additive white Gaussian noise of power spectral density $\frac{N_0}{2} = 10^{-2}$ W/Hz.

PROBLEM 2. A discrete memoryless source produces bits at a rate 10^6 bps. The bits, which are uniformly distributed and i.i.d., are grouped into pairs. Each pair is mapped into a distinct waveform and sent over the AWGN channel of noise power spectral density $\frac{N_0}{2}$. Specifically, the first two bits are mapped into one of the four waveforms shown below with $T_s = 2 \times 10^{-6}$ seconds, the next two bits are mapped onto the same set of waveforms delayed by T_s, etc.



(a) Describe an orthonormal basis for the inner product space $\mathcal{W}$ spanned by $w_i(t)$, $i = 0, 1, 2, 3$ and plot the signal constellation in $\mathbb{R}^n$, where n is the dimensionality of $\mathcal{W}$.

(b) Determine an assignment between pairs of bits and waveforms such that the bit-error probability is minimized and derive an expression for P_b.

(c) Draw a block diagram of the receiver that achieves the above P_b using a single causal filter.

(d) Determine the energy per bit $\mathcal{E}_b$ and the power of the transmitted signal.

PROBLEM 3. m-ary frequency-shift keying (m-FSK) is a signaling method that uses signals of the form

$$w_i(t) = \sqrt{\frac{2\mathcal{E}}{T}} \cos\left(2\pi\left(f_c + i\Delta f\right)t\right) \mathbb{1}_{[0,T)}(t), \quad i = 0, \ldots, m-1,$$

where $\mathcal{E}, T, f_c, \Delta f$ are fixed parameters, with $\Delta f \ll f_c$.

(a) Determine the average energy $\mathcal{E}$. (You can assume $f_c T \in \mathbb{N}$.)

(b) Assuming $f_c T \in \mathbb{N}$, find the smallest value of Δf that makes $w_i(t)$ orthogonal to $w_j(t)$ when $i \neq j$.

(c) In practice the signals $w_i(t)$, $i = 0, \ldots, m-1$ can be generated by changing the frequency of a single oscillator. In passing from one frequency to another, a phase shift θ is introduced. Again, assuming $f_c T \in \mathbb{N}$, determine the smallest value of Δf that ensures orthogonality between $\cos\left(2\pi\left(f_c + i\Delta f\right)t + \theta_i\right)$ and $\cos\left(2\pi\left(f_c + j\Delta f\right)t + \theta_j\right)$ whenever $i \neq j$, regardless of θ_i and θ_j.

(d) Sometimes we do not have complete control over f_c either, in which case it is not possible to set $f_c T \in \mathbb{N}$. Argue that if we choose $f_c T \gg 1$, then for all practical purposes the signals will be orthogonal to one another if the condition found in part (c) is met.

(e) Give an approximate value for the bandwidth occupied by the signal constellation. How does the WT product behave as a function of $k = \log_2(m)$?

PROBLEM 4. Consider using antipodal signaling, i.e. $w_0(t) = -w_1(t)$, to communicate 1 bit across a Rayleigh fading channel that we model as follows. When $w_i(t)$ is transmitted the channel output is

$$R(t) = Aw_i(t) + N(t),$$

where $N(t)$ is white Gaussian noise of power spectral density $\frac{N_0}{2}$ and A is a random variable of probability density function

$$f_A(a) = 2ae^{-a^2}\mathbb{1}\{a \geq 0\}.$$

We assume that, unlike the transmitter, the receiver knows the realization of A. We also assume that the receiver implements a maximum likelihood decision, and that the signal's energy is $\mathcal{E}_b$.

(a) Describe the receiver.

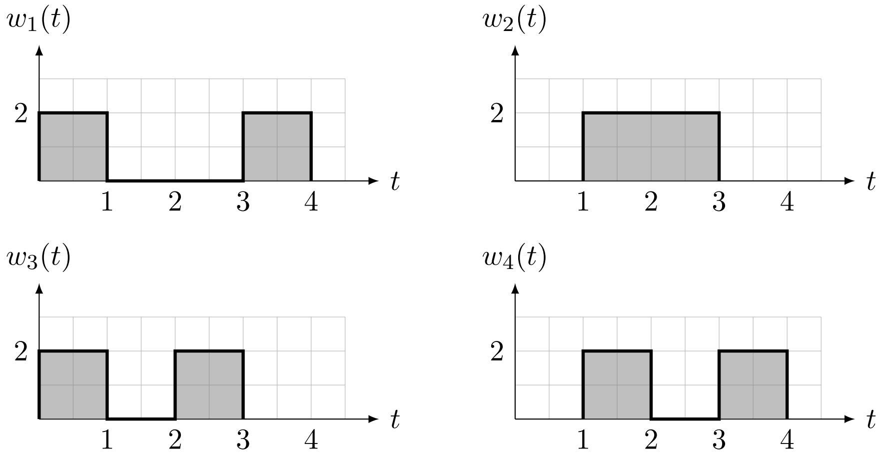
(b) Determine the error probability conditioned on the event $\{A = a\}$.

(c) Determine the unconditional error probability P_f. (The subscript stands for fading.)

(d) Compare P_f to the error probability P_e achieved by an ML receiver that observes $R(t) = mw_i(t) + N(t)$, where $m = \mathbb{E}[A]$. Comment on the different behavior of the two error probabilities. For each of them, find the $\frac{\mathcal{E}_b}{N_0}$ value necessary to obtain the error probability 10^{-5}.

Hint: Use $\frac{1}{2}\exp\left(-\frac{1}{2}x^2\right)$ as an approximation of $Q(x)$.

Problem 5. Consider the signal set shown below. Each signal is equally likely to be chosen for transmission over an AWGN channel with power spectral density $\frac{N_0}{2}$.



(a) Represent the signal set using the four basis signals given by $\psi_1(t) = \psi(t)$, $\psi_2(t) = \psi(t-1)$, $\psi_3(t) = \psi(t-2)$, $\psi_4(t) = \psi(t-3)$, where

$$\psi(t) = \begin{cases} 1 & 0 \leq t \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(b) Use the union bound to find an upper bound to the error probability for the optimal receiver.

(c) Transform the four signals by a translation in order to obtain a minimum energy signal set. Sketch the new signal set $\{\tilde{w}_1(t), \tilde{w}_2(t), \tilde{w}_3(t), \tilde{w}_4(t)\}$.

(d) Use the Gram–Schmidt procedure to find an orthogonal basis for $\{\tilde{w}_1(t), \tilde{w}_2(t), \tilde{w}_3(t), \tilde{w}_4(t)\}$.

(e) Find the exact error probability of an optimal receiver designed for $\{\tilde{w}_1(t), \tilde{w}_2(t), \tilde{w}_3(t), \tilde{w}_4(t)\}$.

(f) Based on your answer to (e), what can you say about the error probability of the receiver in (b)?

Problem 6. This exercise complements what we have learned in Example 4.3 of the book. Consider using the m-PAM constellation

$$\{\pm a, \pm 3a, \pm 5a, \dots, \pm(m-1)a\}$$

to communicate across the discrete-time AWGN channel of noise variance $\sigma^2 = 1$. Our goal is to communicate at some level of reliability, say with error probability $P_e = 10^{-5}$. We are interested in comparing the energy needed by PAM versus the energy needed by a system that operates at channel capacity, namely at $\frac{1}{2}\log_2\left(1 + \frac{\mathcal{E}_s}{\sigma^2}\right)$ bits per channel use.

(a) Using the capacity formula, determine the energy per symbol $\mathcal{E}_s^C(k)$ needed to transmit k bits per channel use. (The superscript C stands for channel capacity.) At any rate below capacity, it is possible to make the error probability arbitrarily small by increasing the codeword length. This implies that there is a way to achieve the desired error probability at energy per symbol $\mathcal{E}_s^C(k)$.

(b) Using single-shot m-PAM, we can achieve an arbitrarily small error probability by making the parameter a sufficiently large. As the size m of the constellation increases, the edge effects become negligible, and the average error probability approaches $2Q\left(\frac{a}{\sigma}\right)$, which is the probability of error conditioned on an interior point being transmitted. Find the numerical value of the parameter a for which $2Q\left(\frac{a}{\sigma}\right) = 10^{-5}$.
Hint: Use $\frac{1}{2}\exp\left(-\frac{1}{2}x^2\right)$ as an approximation of $Q(x)$.

(c) Having fixed the value of a, we can use equation (4.1) of the book to determine the average energy $\mathcal{E}_s^P(k)$ needed by PAM to send k bits at the desired error probability. (The superscript P stands for PAM.) Find and compare the numerical values of $\mathcal{E}_s^P(k)$ and $\mathcal{E}_s^C(k)$ for $k = 1, 2, 4$.

(d) Find $\lim_{n\to\infty} \frac{\mathcal{E}_s^C(k+1)}{\mathcal{E}_s^C(k)}$ and $\lim_{n\to\infty} \frac{\mathcal{E}_s^P(k+1)}{\mathcal{E}_s^P(k)}$.

(e) Comment on PAM's efficiency in terms of energy per bit for small and large values of k. Comment also on the relationship between this exercise and Example 4.3.