Problem 1. Consider the ternary hypothesis testing problem

\[ H_0 : Y = c_0 + Z, \quad H_1 : Y = c_1 + Z, \quad H_2 : Y = c_2 + Z, \]

where \( Y = [Y_1, Y_2]^T \) is the two-dimensional observation vector, \( c_0 = \sqrt{3}[1, 0]^T, \ c_1 = \frac{1}{2}\sqrt{3}[-1, \sqrt{3}]^T, \ c_2 = \frac{1}{2}\sqrt{3}[-1, -\sqrt{3}]^T, \) and \( Z = [Z_1, Z_2]^T \sim \mathcal{N}(0, \sigma_2^2 I_2). \)

(a) Assuming the three hypotheses are equally likely, draw the optimal decision regions in the \((Y_1, Y_2)\) plane.

(b) Assume now that the apriori probabilities for the hypotheses are \( \Pr\{H = 0\} = \frac{1}{2} \), \( \Pr\{H = 1\} = \Pr\{H = 2\} = \frac{1}{4} \). Draw the decision regions in the \((L_1, L_2)\) plane where

\[ L_i := \frac{f_{Y|H}(Y|i)}{f_{Y|H}(Y|0)}, \quad i = 1, 2. \]

Problem 2. Let \( X \sim \mathcal{N}(0, \sigma^2 I_2) \). For each of the three diagrams shown below, express the probability that \( X \) lies in the shaded region. You may use the \( Q \) function when appropriate.

(a) \( x_1 \)

(b) \( x_2 \)

(c) \( x_2 \)

Problem 3. Let \( H \in \{0, 1, 2, 3\} \) and assume that when \( H = i \) you transmit the codeword \( c_i \) shown in the following diagram. Under \( H = i \), the receiver observes \( Y = c_i + Z \).

(a) Draw the decoding regions assuming that \( Z \sim \mathcal{N}(0, \sigma^2 I_2) \) and that \( P_H(i) = 1/4, \quad i \in \{0, 1, 2, 3\} \).

(b) Draw the decoding regions (qualitatively) assuming \( Z \sim \mathcal{N}(0, \sigma^2 I_2) \) and \( P_H(0) = P_H(2) > P_H(1) = P_H(3) \). Justify your answer.

(c) Assume again that \( P_H(i) = 1/4, \quad i \in \{0, 1, 2, 3\} \) and that \( Z \sim \mathcal{N}(0, K) \), where \( K = \begin{pmatrix} \sigma^2 & 0 \\ 0 & 4\sigma^2 \end{pmatrix} \). How do you decode now?

Problem 4. The following problem relates to the design of multi-antenna systems. Consider the binary equiprobable hypothesis testing problem:

\[ H = 0 : Y_1 = A + Z_1, \quad Y_2 = A + Z_2 \]
\[ H = 1 : Y_1 = -A + Z_1, \quad Y_2 = -A + Z_2 \]

where \( Z_1, \ Z_2 \) are independent Gaussian random variables with different variances \( \sigma_1^2 \neq \sigma_2^2 \), that is, \( Z_1 \sim \mathcal{N}(0, \sigma_1^2) \) and \( Z_2 \sim \mathcal{N}(0, \sigma_2^2) \). \( A > 0 \) is a constant.
(a) Show that the decision rule that minimizes the probability of error (based on the observable \( Y_1 \) and \( Y_2 \)) can be stated as

\[
\sigma_2^2 y_1 + \sigma_1^2 y_2 \geq \frac{\sigma^2}{1}
\]

(b) Draw the decision regions in the \((Y_1, Y_2)\) plane for the special case where \( \sigma_1 = 2\sigma_2 \).

(c) Evaluate the probability of the error for the optimal detector as a function of \( \sigma_1^2, \sigma_2^2 \) and \( A \).

**Problem 5.** The process of storing and retrieving binary data on a thin-film disk can be modeled as transmitting binary symbols across an additive white Gaussian noise channel where the noise \( Z \) has a variance that depends on the transmitted (stored) binary symbol \( X \). The noise has the following input-dependent density:

\[
f_Z(z) = \begin{cases} 
\frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{z^2}{2\sigma_1^2}} & \text{if } X = 1 \\
\frac{1}{\sqrt{2\pi\sigma_0^2}} e^{-\frac{z^2}{2\sigma_0^2}} & \text{if } X = 0,
\end{cases}
\]

where \( \sigma_1 > \sigma_0 \). The channel inputs are equally likely.

(a) On the same graph, plot the two possible output probability density functions. Indicate, qualitatively, the decision regions.

(b) Determine the optimal receiver in terms of \( \sigma_0 \) and \( \sigma_1 \).

(c) Write an expression for the error probability \( P_e \) as a function of \( \sigma_0 \) and \( \sigma_1 \).

**Problem 6.** The following two signal constellations are used to communicate across an additive white Gaussian noise channel. Let the noise variance be \( \sigma^2 \). Each point represents a codeword \( c_i \) for some \( i \). Assume each codeword is used with the same probability.

(a) For each signal constellation, compute the average probability of error \( P_e \) as a function of the parameters \( a \) and \( b \), respectively.

(b) For each signal constellation, compute the average energy per symbol \( E \) as a function of parameters \( a \) and \( b \), respectively:

\[
E = \frac{1}{16} \sum_{i=1}^{16} P_H(i) \| c_i \|^2
\]

(c) Plot \( P_e \) versus \( \frac{E}{\sigma^2} \) for both signal constellations and comment.