Isoperimetric inequalities

Theorem ("the" isoperimetric inequality)

Among bodies of given volume, Eucliden balls have the least surface area.

Proof:

unit ball

Let C be a compact set in $\mathbb{R}^n$ with $\mathrm{vol}(C) = \mathrm{Vol}(B_2^n)$.

Surface area: $\sigma(\partial C) = \lim_{\varepsilon \to 0} \dfrac{\mathrm{vol}(C+\varepsilon B_2^n) - \mathrm{vol}(C)}{\varepsilon}$

Additive BM inequality: $\mathrm{Vol}(C+\varepsilon B_2^n)^{1/n} \geq (1-\varepsilon)\,\mathrm{vol}\left(\frac{C}{1-\varepsilon}\right)^{1/n} + \varepsilon\, \mathrm{vol}(B_2^n)^{1/n}$

So $\mathrm{Vol}(C+\varepsilon B_2^n) - \mathrm{Vol}(C) \geq \left(\mathrm{vol}(C)^{1/n} + \varepsilon\, \mathrm{vol}(B_2^n)^{1/n}\right)^n - \mathrm{Vol}(C)$

$\geq \cancel{\mathrm{vol}\, C} + n\, \mathrm{vol}(C)^{\frac{n-1}{n}} \varepsilon\, \mathrm{vol}(B_2^n)^{\frac{1}{n}} - \cancel{\mathrm{Vol}\, C} = n \varepsilon\, \mathrm{vol}(B_2^n)$

So $\sigma(\partial C) \geq n\, \mathrm{vol}(B_2^n) = \sigma(\partial B_2^n)$ #

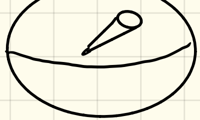
Explanation of $\sigma(\partial B_2^n) = n\,\mathrm{Vol}(B_2^n)$: cone argument

If B is the base (dim $n-1$) of a cone C of height 1

Then $\mathrm{Vol}(C) = \int_0^1 dt\,(t)^{n-1}\cdot\sigma(B) = \frac{\sigma(B)}{n}$.

Dividing a sphere into smaller & smaller cones, we obtain:

$$\mathrm{Vol}(B_2^n) = \frac{\sigma(\partial B_2^n)}{n}$$

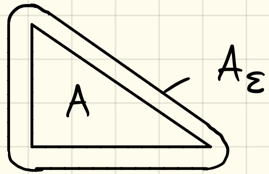


Alternative way of stating the isoperimetric inequality:

For $x \in \mathbb{R}^n$ & $A \subset \mathbb{R}^n$, define $d(x,A) = \inf\{\|x-y\|_2 : y \in A\}$

For $\varepsilon > 0$ & $A \subset \mathbb{R}^n$, define $A_\varepsilon = \{x \in \mathbb{R}^n : d(x,A) \le \varepsilon\}$

$= A + \varepsilon B_2^n$ = "ε-neighborhood of A"



Isoperimetric inequality (v2)

Given $A \subset \mathbb{R}^n$ compact & non-empty, it holds $\forall \varepsilon > 0$ that $\mathrm{Vol}(A_\varepsilon) \geq \mathrm{Vol}(B_\varepsilon)$ where B is the Euclidean ball with the same volume as A.

So Euclidean balls are the least expanding sets.

Proof:

$$\mathrm{Vol}(A_\varepsilon) = \mathrm{Vol}(A + \varepsilon B) = \mathrm{Vol}\left((1-\varepsilon)\frac{A}{1-\varepsilon} + \varepsilon B\right)$$

$$\geq \left((1-\varepsilon)\,\mathrm{Vol}\left(\frac{A}{1-\varepsilon}\right)^{1/n} + \varepsilon\,(\mathrm{Vol}\,B)^{1/n}\right)^n \text{ by BM inequality}$$

$$= \left(\mathrm{Vol}(A)^{1/n} + \varepsilon\,\mathrm{Vol}(B)^{1/n}\right)^n$$

$$= \left(\mathrm{Vol}(B)^{1/n} + \varepsilon\,\mathrm{Vol}(B)^{1/n}\right)^n$$

$$= (1+\varepsilon)^n \cdot \mathrm{Vol}(B) = \mathrm{Vol}(B_\varepsilon)$$

#

Isoperimetric inequalities in different settings

1. Sphere $S^{n-1} \subset \mathbb{R}^n$

distance d = geodesic distance on the sphere or
Euclidean distance inherited from $\mathbb{R}^n$

$\sigma(A)$ = spherical measure of $A \subset S^{n-1}$

Theorem (Lévy, without proof)

Given $A \subset S^{n-1}$ compact & non-empty, it holds $\forall \varepsilon > 0$ that
$\sigma(A_\varepsilon) \geq \sigma(C_\varepsilon)$ where C is a cap of same volume.

So caps are the least expanding sets in this setting.

Corollary

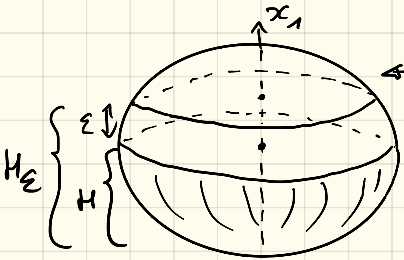
Let $A \subset S^{n-1}$ be a closed set such that $\sigma(A) = \frac{1}{2}$ (we consider here the spherical measure normalized to 1).
Then $\forall\, 0 < \varepsilon \leq \frac{1}{\sqrt{2}}$, $\sigma(A_\varepsilon) \geq 1 - \exp(-\varepsilon^2 n/2)$

That is, as $n \to \infty$, nearly all the area of the sphere is contained in the set A_ε ! (even though A is "any" set such that $vol(A) = \frac{1}{2}$)

Proof:

By Levy's theorem, $\sigma(A_\varepsilon) \geq \sigma(H_\varepsilon)$ where H is the southern hemisphere:

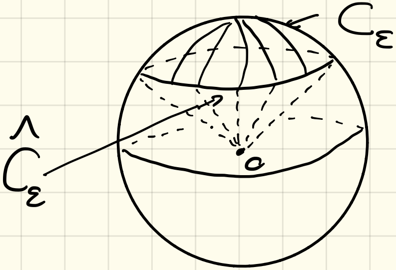
(⚠ valid for ε small only)



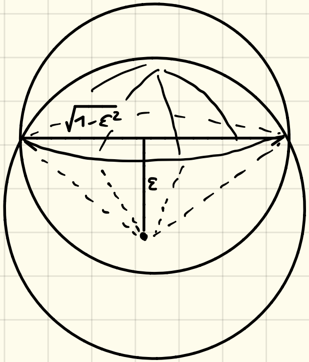
cap $C_\varepsilon = H_\varepsilon^c$
$= \{x \in S^{n-1} : x_1 \geq \varepsilon\}$

So $\sigma(A_\varepsilon) \geq 1 - \sigma(C_\varepsilon)$.

To evaluate the (normalized) surface of the cap C_ε let us observe that it is equal to the (normalized) volume of the cone $\hat{C}_\varepsilon$ from the center of the sphere to C_ε:



And when $\varepsilon \leq \frac{1}{\sqrt{2}}$, this cone $\hat{C}_\varepsilon$ is contained in a sphere of radius $\sqrt{1-\varepsilon^2}$:



So $\mathrm{Vol}(\hat{C}_\varepsilon) \leq \sqrt{1-\varepsilon^2}^{\,n}$

(again the normalized volume)

Finally, we obtain

$$\sigma(A_\varepsilon) \geq 1 - \mathrm{vol}(\hat{C}_\varepsilon) \geq 1 - \left(\sqrt{1-\varepsilon^2}\right)^n$$
$$= 1 - (1-\varepsilon^2)^{n/2} \underset{(1-x \leq e^{-x})}{\geq} 1 - \exp(-n\varepsilon^2/2) \qquad \#$$

<u>NB</u>: • When $A = H$, this is saying that $\sigma(H_\varepsilon) \geq 1 - \exp\left(-\frac{n\varepsilon^2}{2}\right)$ and by a symmetric argument, we obtain

$$\sigma(E_\varepsilon) \geq 1 - 2\exp\left(-\frac{n\varepsilon^2}{2}\right)$$

where E is the equator $\left(= \{x \in S^{n-1} : x_1 = 0\}\right)$

(and E_ε is the "small" strip of width ε around it)

• Two uniform & independent vectors u, v on S^{n-1} satisfy

$$\mathbb{P}(u^T v \geq \varepsilon) = \sigma(C_\varepsilon) \leq e^{-n\varepsilon^2/2} \text{ : they are almost orthogonal!}$$

Alternate version and proof of the corollary of Lévy's theorem:

Thm

Let $A \subset S^{n-1}$ be s.t. (normalized) $\sigma(A) \geq \frac{1}{2}$.

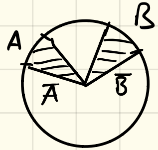
Then for any $\varepsilon > 0$, $\sigma(A_\varepsilon) \geq 1 - 2\exp(-n\varepsilon^2/4)$

Proof:

- Let $B = \{ y \in S^{n-1} : d(y, A) \geq \varepsilon \}$, $d =$ Euclidean distance

$\forall x \in A$ & $y \in B$ $\quad \left\| \frac{x+y}{2} \right\|_2^2 = \frac{\|x\|_2^2}{2} + \frac{\|y\|_2^2}{2} - \frac{\|x-y\|_2^2}{4} \leq 1 - \frac{\varepsilon^2}{4}$, thus

$\left\| \frac{x+y}{2} \right\|_2 \leq 1 - \frac{\varepsilon^2}{8}$



- Let $\bar{A} = \operatorname{conv}(A \cup \{0\})$, $\bar{B} = \operatorname{conv}(B \cup \{0\})$

Then it also holds that $\forall \bar{x} \in A$ & $\bar{y} \in B$:

$$\left\| \frac{\bar{x}+\bar{y}}{2} \right\|_2 = \left\| \frac{\alpha x + \beta y}{2} \right\|_2 = \left\| \alpha \left(\frac{x+y}{2} \right) + (\beta - \alpha) \frac{y}{2} \right\|_2 \leq \alpha \left(1 - \frac{\varepsilon^2}{8} \right) + \frac{\beta - \alpha}{2} \leq 1 - \frac{\varepsilon^2}{8}$$

The above computation shows that $\frac{\bar{A}+\bar{B}}{2} \subset \left(1-\frac{\varepsilon^2}{8}\right)\cdot B_2^n$.

Observe also that $\underset{\substack{\text{normalized} \\ \sigma(S^{n-1})=1}}{\sigma(A)} = \underset{\substack{\text{normalized} \\ \mathrm{vol}(B_2^n)=1}}{\mathrm{vol}(\bar{A})}$.

Apply now multiplicative BM inequality to $\bar{A}$ & $\bar{B}$:

$$\left(1-\frac{\varepsilon^2}{8}\right)^n \underbrace{\mathrm{vol}(B_2^n)}_{=1} \geq \mathrm{vol}\left(\frac{\bar{A}+\bar{B}}{2}\right) \geq \left(\mathrm{vol}\,\bar{A}\right)^{1/2}\cdot\left(\mathrm{vol}\,\bar{B}\right)^{1/2}$$

Hence, $1-\sigma(A_\varepsilon) = \sigma(B) = \mathrm{vol}(\bar{B}) \leq \dfrac{\left(1-\frac{\varepsilon^2}{8}\right)^{2n}}{\mathrm{vol}(\bar{A})} = \dfrac{\left(1-\frac{\varepsilon^2}{8}\right)^{2n}}{\sigma(A)}$

$$\leq 2\left(1-\frac{\varepsilon^2}{8}\right)^{2n} \leq 2\exp(-n\varepsilon^2/4) \quad \#$$

<u>Note</u>: This proof extends to strictly convex surfaces

Here is (yet) another reformulation of Levy's theorem:

Theorem (always considering the normalized spherical measure)

Let $f : S^{n-1} \to \mathbb{R}$ be a 1-Lipschitz function

($i.e.$ $|f(x) - f(y)| \le \|x-y\|_2 \quad \forall x,y \in S^{n-1}$)

Then $\sigma(\{f > M_f + \varepsilon\}) \le \exp(-n\varepsilon^2/2) \quad \forall 0 < \varepsilon \le \frac{1}{\sqrt{2}}$

where M_f is the median of f, i.e., the real number such that both $\sigma(\{f \ge M_f\}) \ge \frac{1}{2}$ & $\sigma(\{f \le M_f\}) \ge \frac{1}{2}$.

Note that similarly, we can show that

$$\sigma(\{f < M_f - \varepsilon\}) \le \exp(-n\varepsilon^2/2) \quad \forall 0 < \varepsilon \le \frac{1}{\sqrt{2}}$$

This is saying that if X is a point drawn uniformly at random on the sphere, then $f(X)$ is ε-close to its median with high probability (as the dimension n gets large).

Proof of the theorem:

Let $A = \{ f \leq M_f \}$: then $x \in A_\varepsilon$ iff $\exists y \in A$ s.t. $\|x-y\|_2 \leq \varepsilon$

But in this case, $|f(x) - f(y)| \leq \|x-y\|_2 \leq \varepsilon$ by the Lipschitz prop.

So $f(x) \leq f(y) + \varepsilon \leq M_f + \varepsilon$ as $y \in A$

So $\sigma(\{ f \leq M_f + \varepsilon \}) \geq \sigma(A_\varepsilon) \geq 1 - \exp(-n\varepsilon^2/2)$

[NB: $\sigma(A) \geq \frac{1}{2}$ in general, but this can only increase $\sigma(A_\varepsilon)$]

i.e. $\sigma(\{ f > M_f + \varepsilon \}) \leq \exp(-n\varepsilon^2/2)$. #

2. Gaussian measure on $\mathbb{R}^n$

d = Euclidean distance

$$\nu d(A) = \int_A dx \underbrace{\frac{1}{(2\pi)^{n/2}} \cdot \exp\left(-\frac{\|x\|_2^2}{2}\right)}_{\text{standard Gaussian pdf}}$$

Theorem (Borell, without proof)

If $A \subset \mathbb{R}^n$ is closed and such that $\nu d(A) = \frac{1}{2}$, then it holds $\forall \varepsilon > 0$ that $\nu d(A_\varepsilon) \geq \nu d(H_\varepsilon)$, where H is the half space $H = \{x \in \mathbb{R}^n : x_1 \leq 0\}$

So half spaces are the least expanding sets here.

Link between 1 (sphere) & 2 (Gaussian)

Consider $X_1 \dots X_n$ iid $\sim N(0,1)$: their joint pdf is

$$p_X(x) = \frac{1}{(2\pi)^{n/2}} \cdot \exp\left(-\frac{\|x\|_2^2}{2}\right)$$

It turns out that:

- the random vector $X / \|X\|_2 = \frac{1}{\|X\|_2}(X_1, \dots, X_n)$ is uniformly distributed on the sphere S^{n-1}
- $\|X\|_2 = \sqrt{X_1^2 + \dots + X_n^2}$ concentrates around the value $\sqrt{n}$ as $n \to \infty$ (so the random vector X itself concentrates on the sphere of radius $\sqrt{n}$)

Corollary of Borell's theorem

Let $A \subset \mathbb{R}^n$ be a closed set such that $\mathrm{Vol}(A) = \frac{1}{2}$.
Then $\mathrm{Vol}(A_\varepsilon) \geq 1 - \exp(-\varepsilon^2/2) \quad \forall \varepsilon > 0$

Proof

By Borell's thm, $\mathrm{Vol}(A_\varepsilon) \geq \mathrm{Vol}(H_\varepsilon) = \mathbb{P}(\{x_1 \leq \varepsilon\})$
$= 1 - \mathbb{P}(\{x_1 > \varepsilon\}) \geq 1 - \exp(-\varepsilon^2/2)$ (Chebyshev) #

NB: This is saying that $\mathrm{vol}(A_{\sqrt{n}\varepsilon}) \xrightarrow[n\to\infty]{} 1$
(even though $\mathrm{vol}(A) = \frac{1}{2}$)

(do not forget that we are considering the whole space $\mathbb{R}^n$ here!)

Here is an alternate derivation of the above corollary (up to factors 2) that does <u>not</u> require using the isoperimetric inequality, but uses instead the Prékopa-Leindler inequality.

<u>Lemma</u>

Let X be a Gaussian vector with joint pdf p_X as before. and $A \subset \mathbb{R}^n$. Then $\mathbb{E}\left(\exp\left(d(X,A)^2/4\right)\right) \leq \frac{1}{\mathbb{P}(X \in A)}$.

<u>Consequence</u>

If $\mathbb{P}(X \in A) = \frac{1}{2}$, then

$$\mathbb{P}\big(\underbrace{d(X,A) > \varepsilon}_{\text{i.e. } X \notin A_\varepsilon}\big) \underset{\text{Chebyshev}}{\leq} \frac{\mathbb{E}\left(\exp\left(d(X,A)^2/4\right)\right)}{\exp\left(\varepsilon^2/4\right)} \leq 2 \cdot \exp\left(-\varepsilon^2/4\right)$$

Proof of the lemma:

Let $f(x) = \exp(d(x,A)^2/4)\, p(x)$, $g(x) = 1_A(x)\, p(x)$, $m(x) = p(x)$:

$$\mathbb{E}\left(\exp(d(X,A)^2/4)\right) = \int_{\mathbb{R}^n} dx\, f(x), \quad \mathbb{P}(X \in A) = \int_{\mathbb{R}^n} dx\, g(x), \quad 1 = \int_{\mathbb{R}^n} dx\, m(x)$$

PL inequality with $\lambda = \frac{1}{2}$: $\left(\int_{\mathbb{R}^n} dx\, f(x)\right) \cdot \left(\int_{\mathbb{R}^n} dx\, g(x)\right) \leq \left(\int_{\mathbb{R}^n} dx\, m(x)\right)^2 = 1$

i.e. what we want!

Condition to be checked:

$$f(x) \cdot g(y) \leq m\left(\frac{x+y}{2}\right)^2 \quad \forall x, y \in \mathbb{R}^n$$

Indeed: if $y \notin A$, there is nothing to prove, so assume $y \in A$:

$$f(x)\, g(y) = \frac{1}{(2\pi)^n} \exp\left(-\frac{d(x,A)^2}{4} - \frac{\|x\|^2}{2} - \frac{\|y\|^2}{2}\right)$$

$$\leq \frac{1}{(2\pi)^n} \exp\left(-\frac{\|x-y\|^2}{4} - \frac{\|x\|^2}{2} - \frac{\|y\|^2}{2}\right) = \frac{1}{(2\pi)^n} \exp\left(-\frac{\|x+y\|^2}{4}\right)$$

$$= m\left(\frac{x+y}{2}\right)^2 \quad \#$$