Convex geometry

Plan

Brunn-Minkowski inequality

Grünbaum's theorem

Isoperimetric inequalities

Measure concentration

Efficient sampling of a convex body

Approximate center of gravity of a convex body

Rounding of a convex body (inertia ellipsoid)

Brunn - Minkowski inequality

1. **Additive version:** (NB: $A+B = \{x+y : x \in A, y \in B\}$)

   A, B non-empty compact sets in $\mathbb{R}^n$

   $$\Rightarrow \mathrm{vol}((1-\lambda)A + \lambda B)^{1/n} \geq (1-\lambda)\,\mathrm{vol}(A)^{1/n} + \lambda\,\mathrm{vol}(B)^{1/n} \qquad 0 \leq \lambda \leq 1$$

2. **Multiplicative version:**

   A, B compact sets in $\mathbb{R}^n$

   $$\Rightarrow \mathrm{vol}((1-\lambda)A + \lambda B) \geq \mathrm{vol}(A)^{1-\lambda} \cdot \mathrm{vol}(B)^{\lambda} \qquad 0 \leq \lambda \leq 1$$

Remarks:

- "non-empty" needed in 1 because A empty $\Rightarrow (1-\lambda)A + \lambda B$ empty
- also, the multiplicative version is "dimension free" (no $\frac{1}{n}$ factor)

Proposition : $1 \Longleftrightarrow 2$

a) $\underline{1 \Longrightarrow 2}$: AM-GM ineq : $(1-\lambda)a+\lambda b \geq a^{1-\lambda} b^{\lambda}$ $\forall a,b \geq 0,\ 0 \leq \lambda \leq 1$

So by 1, we have:

$$\mathrm{vol}((1-\lambda)A+\lambda B)^{1/n} \geq (1-\lambda)(\mathrm{vol}\,A)^{1/n} + \lambda(\mathrm{vol}\,B)^{1/n} \geq (\mathrm{vol}\,A)^{\frac{1-\lambda}{n}}(\mathrm{vol}\,B)^{\frac{\lambda}{n}} \checkmark \ \left(\begin{array}{c}\text{take}\\ \text{power } n\end{array}\right)$$

(and we can allow now A or B empty)

b) $\underline{2 \Longrightarrow 1}$: • If $\mathrm{vol}\,A=0$ or $\mathrm{vol}\,B=0$, nothing to prove.

• Assume $\mathrm{vol}(A)>0$ & $\mathrm{vol}(B)>0$, and let $A' = \dfrac{A}{(\mathrm{vol}\,A)^{1/n}}$ $B' = \dfrac{B}{(\mathrm{vol}\,B)^{1/n}}$

and $\lambda' = \dfrac{\lambda(\mathrm{vol}\,B)^{1/n}}{(1-\lambda)(\mathrm{vol}\,A)^{1/n}+\lambda(\mathrm{vol}\,B)^{1/n}} \in [0,1]$. Then by 2, we have

$$1 = \mathrm{vol}(A')^{1-\lambda'}\mathrm{vol}(B')^{\lambda'} \leq \mathrm{vol}((1-\lambda')A'+\lambda' B') = \mathrm{vol}\left(\frac{(1-\lambda)A+\lambda B}{(1-\lambda)(\mathrm{vol}\,A)^{1/n}+\lambda(\mathrm{vol}\,B)^{1/n}}\right)$$

so $\mathrm{vol}((1-\lambda)A+\lambda B) \geq \left((1-\lambda)(\mathrm{vol}\,A)^{1/n}+\lambda(\mathrm{vol}\,B)^{1/n}\right)^n \checkmark$ (take power $1/n$)

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Proof strategy

A. proof of the additive BM inequality in dimension 1

B. used to prove a generalization of the multiplicative BM inequality in dimension 1, named the Prékopa-Leindler inequality

C. recursive proof of the n-dim. PL inequality, (over the dimension n) implying the multiplicative BM ineq, implying the additive BM ineq.

(length in dimension 1)
$\quad \,\, \parallel$

A. <u>To be shown</u>: $\operatorname{vol}(A+B) \geq \operatorname{vol} A + \operatorname{vol} B$ for A, B compact $\subset \mathbb{R}$ non-empty

Observe $\operatorname{vol}(A+x) = \operatorname{vol} A \quad \forall x \in \mathbb{R}$, so we may choose A with right-most point $=0$, B with left-most point $=0$ so that A, B are almost disjoint sets, $A+B$ contains both $\underset{\subset A+\{0\}}{A}$ and $\underset{\subset \{0\}+B}{B}$, so $\operatorname{vol}(A+B) \geq \operatorname{vol} A + \operatorname{vol} B$.

✓ (part A)

B. <u>One-dimensional Prékopa-Leindler inequality</u>:

Let $f, g, m : \mathbb{R} \to \mathbb{R}$ be non-negative measurable fns.
If $m((1-\lambda)x + \lambda y) \geq f(x)^{1-\lambda} \cdot g(y)^{\lambda} \quad \forall x, y \in \mathbb{R},\ 0 \leq \lambda \leq 1$
then
$$\int_{\mathbb{R}} m(x)\, dx \geq \left(\int_{\mathbb{R}} f(x)\, dx\right)^{1-\lambda} \cdot \left(\int_{\mathbb{R}} g(x)\, dx\right)^{\lambda}$$

Note: **PL $\Rightarrow$ mult. BM** (same argument in dimension n):

Given A, B, define $f(x) = 1_A(x)$, $g(x) = 1_B(x)$ and

$$m(x) = 1_{(1-\lambda)A+\lambda B}(x)$$

PL inequality states: $\int_{\mathbb{R}} m(x)\,dx \geq \left(\int_{\mathbb{R}} f(x)dx\right)^{1-\lambda} \cdot \left(\int_{\mathbb{R}} g(x)\,dx\right)^{\lambda}$

i.e. $\mathrm{vol}((1-\lambda)A+\lambda B) \geq (\mathrm{vol}\,A)^{1-\lambda} \cdot (\mathrm{vol}\,B)^{\lambda}$ i.e. what we want

To be checked still: $m((1-\lambda)x+\lambda y) \geq f(x)^{1-\lambda} \cdot g(y)^{\lambda}$

$\forall x, y \in \mathbb{R},\ 0 \leq \lambda \leq 1$?

For 0-1 valued functions, this amounts to show that:

if $f(x) = 1$ and $g(y) = 1$, then $m((1-\lambda)x+\lambda y) = 1$

equiv. to: if $x \in A$ and $y \in B$, then $(1-\lambda)x+\lambda y \in (1-\lambda)A+\lambda B$ ✓

Proof of the one-dim. PL inequality:

- Assume both $\int f < +\infty$ and $\int g < +\infty$ (o.w. nothing to prove)
- $f \geq 0$, so $\int_{\mathbb{R}} f(x)\,dx = \int_{\mathbb{R}} \left(\int_0^{f(x)} 1\,dt \right) dx = \int_{\mathbb{R}} \int_0^{\infty} 1_{\{f(x) \geq t\}}\,dt\,dx$

$$\overset{\text{(Fubini)}}{=} \int_0^{\infty} \int_{\mathbb{R}} 1_{\{f(x) \geq t\}}\,dx\,dt = \int_0^{\infty} \operatorname{vol}(\{f \geq t\})\,dt$$

- same for g and m
- Observe that if $f(x) \geq t$ & $g(y) \geq t$, then $m((1-\lambda)x + \lambda y) \geq t$ (by assumption)

so $\{m \geq t\} \supset (1-\lambda)\{f \geq t\} + \lambda\{g \geq t\}$

- So $\int_{\mathbb{R}} m(x)\,dx = \int_0^{\infty} \operatorname{vol}(\{m \geq t\})\,dt$

$$\geq \int_0^{\infty} \operatorname{vol}\left((1-\lambda)\{f \geq t\} + \lambda\{g \geq t\}\right) dt$$

$\cdot$ So $\displaystyle\int m \geq (1-\lambda)\cdot\int_0^\infty \mathrm{Vol}(\{f\geq t\})\,dt + \lambda\cdot\int_0^\infty \mathrm{vol}(\{g\geq t\})\,dt$

by the one-dim. additive B-M ineq.

$$= (1-\lambda)\cdot\int f + \lambda\cdot\int g$$

$$\geq \left(\int f\right)^{1-\lambda}\cdot\left(\int g\right)^{\lambda}$$ by the AM-GM inequality ✓

(part B)

C. We now set out to prove recursively the <u>multidimensional PL inequality</u>:

Let $f, g, m : \mathbb{R}^n \to \mathbb{R}$ be measurable fns s.t.

$m((1-\lambda)x + \lambda y) \geq f(x)^{1-\lambda}\, g(y)^{\lambda} \quad \forall\, x, y \in \mathbb{R}^n,\ 0\leq\lambda\leq 1$

Then $\displaystyle\int_{\mathbb{R}^n} m \geq \left(\int_{\mathbb{R}^n} f\right)^{1-\lambda}\cdot\left(\int_{\mathbb{R}^n} g\right)^{\lambda}$

Proof:

- The base case $n=1$ has just been proven.
- Assume now the inequality holds for dimension $n-1$:

For $f: \mathbb{R}^n \to \mathbb{R}$ & $z \in \mathbb{R}$, write $f_z: \mathbb{R}^{n-1} \to \mathbb{R}$ as $f_z(x) = f(\underset{\text{dim } 1}{z}, \underset{\text{dim } n-1}{x})$

and $F_z = \int_{\mathbb{R}^{n-1}} f_z(x)\,dx$. The PL ineq. can be rewritten as:

$$\int_{\mathbb{R}} M_z\,dz \geq \left(\int_{\mathbb{R}} F_z\,dz\right)^{1-\lambda} \left(\int_{\mathbb{R}} G_z\,dz\right)^{\lambda}$$

This will follow if we can prove that

$$M_{(1-\lambda)a+\lambda b} \geq F_a^{1-\lambda} \cdot G_b^{\lambda} \qquad \forall a,b \in \mathbb{R},\ 0 \leq \lambda \leq 1$$

Our assumption may be rewritten as:

$$m\big(\underset{\text{dim } 1}{(1-\lambda)a+\lambda b}, \underset{\text{dim } n-1}{(1-\lambda)x+\lambda y}\big) \geq \big(f(a,x)\big)^{1-\lambda} \cdot \big(g(b,y)\big)^{\lambda}$$

i.e. $m_{(1-\lambda)a+\lambda b}\big((1-\lambda)x+\lambda y\big) \geq \big(f_a(x)\big)^{1-\lambda} \cdot \big(g_b(y)\big)^{\lambda} \quad \forall x,y \in \mathbb{R}^{n-1}$

By the induction hypothesis, we have

$$\int_{\mathbb{R}^{n-1}} m_{(1-\lambda)a+\lambda b} \geq \Big(\int_{\mathbb{R}^{n-1}} f_a\Big)^{1-\lambda} \Big(\int_{\mathbb{R}^{n-1}} g_b\Big)^{\lambda}$$

This is exactly what we needed:

$$M_{(1-\lambda)a+\lambda b} \geq F_a^{1-\lambda} \cdot G_b^{\lambda} \quad \checkmark \text{ (part c)} \quad \#$$

NB: equality in BM occurs when A is convex et $B = cA + d$ (note $(1-\lambda)A + \lambda A \subset A$ if A convex)

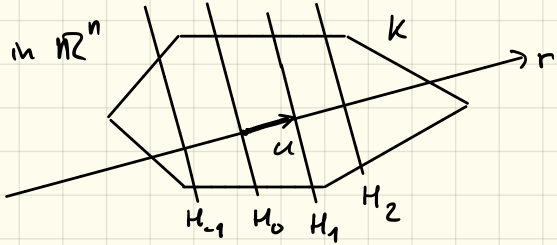
Brunn's theorem

Let K be a convex body in $\mathbb{R}^n$ and $u \in \mathbb{R}^n$

For $r \in \mathbb{R}$, define $H_r = \{x \in \mathbb{R}^n : u^T x = r\}$

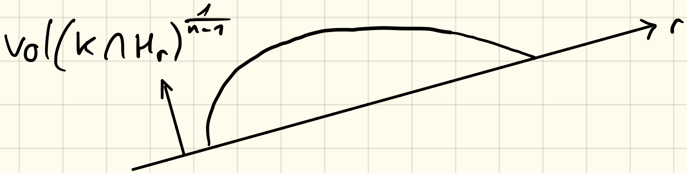
Then the function $r \mapsto \mathrm{vol}(K \cap H_r)^{\frac{1}{n-1}}$ is concave (on its support)

Illustration:



$K \cap H_r$ = section of K

$\mathrm{vol}(K \cap H_r)^{\frac{1}{n-1}} \doteq$ radius



NB: Without the $\frac{1}{n-1}$ power, this does not hold for $n \geq 3$.

If K is a cone in $\mathbb{R}^3$: $\mathrm{vol}(K \cap H_r) \doteq r^2$, $\mathrm{vol}(K \cap H_r)^{\frac{1}{2}} \doteq r$ (worst case)

Proof:

Brunn's theorem is a direct consequence of the additive BM inequality (in $\mathbb{R}^{n-1}$). Consider $A = K \cap H_{r_1}$, $B = K \cap H_{r_2}$:

($r_1, r_2 \in \mathbb{R}$, $0 \leq \lambda \leq 1$)

- K convex $\Rightarrow (1-\lambda) A + \lambda B \subset K$
- $(1-\lambda) H_{r_1} + \lambda H_{r_2} = \{ z = (1-\lambda) x + \lambda y : u^T x = r_1, u^T y = r_2 \}$

$$= \{ z : u^T z = (1-\lambda) r_1 + \lambda r_2 \} = H_{(1-\lambda) r_1 + \lambda r_2}$$

So $\mathrm{Vol}(K \cap H_{(1-\lambda) r_1 + \lambda r_2})^{\frac{1}{n-1}} \geq \mathrm{Vol}((1-\lambda) A + \lambda B)^{\frac{1}{n-1}}$

$$\underset{\text{add. BM}}{\geq} (1-\lambda) (\mathrm{Vol}\, A)^{\frac{1}{n-1}} + \lambda (\mathrm{Vol}\, B)^{\frac{1}{n-1}} = (1-\lambda) \mathrm{Vol}(K \cap H_{r_1})^{\frac{1}{n-1}} + \lambda \mathrm{Vol}(K \cap H_{r_2})^{\frac{1}{n-1}}$$

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Note: From the multiplicative BM inequality, we would get similarly: $r \longmapsto \log \mathrm{Vol}(K \cap H_r)$ is concave (on its support)

Grünbaum's theorem

Let $K \subset \mathbb{R}^n$ be a centered convex set : $\int_K x \, dx = 0$

Then for any $v \in \mathbb{R}^n$ with $v \neq 0$, we have

$$\mathrm{vol}(K \cap \{x \in \mathbb{R}^n : v^T x \geq 0\}) \geq \frac{1}{e} \mathrm{vol}(K)$$

Proof strategy

A. Symmetrization of K along the axis v

B. "Conification" of K along the same axis
(showing that cone = worst case)

C. Direct computation

Without loss of generality, assume $v = e_1$.

A. For $z \in \mathbb{R}$, define $K_z = \{x \in K : x_1 = z\}$ and K'_z the $n-1$ dimensional ball with $\mathrm{vol}(K'_z) = \mathrm{vol}(K_z)$.

Illustration:

Then K' is centered (clear) and convex.

Proof: To be shown: $z \longmapsto r(z) = \mathrm{radius}(K'_z)$ is concave

$$\text{But } r(z) = \left(\frac{\mathrm{vol}(K_z)}{v_{n-1}}\right)^{\frac{1}{n-1}} \text{ where } v_{n-1} = \mathrm{vol}(B_{n-1})$$

unit ball in dim $n-1$

So Brunn's thm allows to conclude. ✓ (part A)

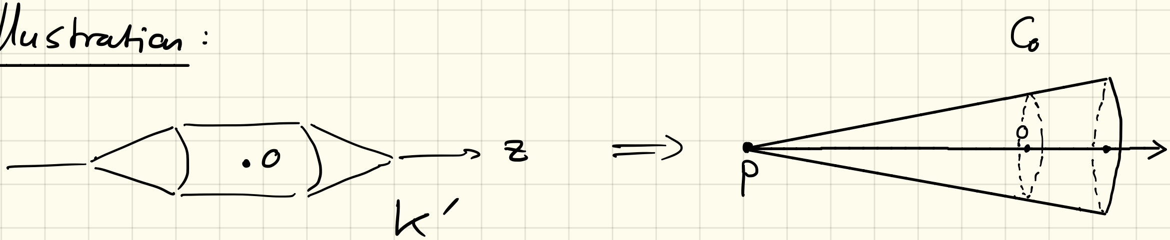
B. We now transform K' into a cone C.

<u>Properties of C</u>:

- $K_0' = C_0$ (same central slice)
- <u>Notation</u>: $X_{\leq z} = \{ x \in X : x_1 \leq z \}$

  The cone C terminates at a point p "on the left" such that $\mathrm{Vol}(K'_{\leq 0}) = \mathrm{Vol}(C_{\leq 0})$
- The cone extends "to the right" such that $\mathrm{Vol}(K'_{\geq 0}) = \mathrm{Vol}(C_{\geq 0})$

<u>Illustration</u>:



Claim

For any K' such that $K'_0 = C_0$, $\mathrm{vol}(K'_{\leq 0}) = \mathrm{Vol}(C_{\leq 0})$ and $\mathrm{vol}(K'_{\geq 0}) = \mathrm{Vol}(C_{\geq 0})$, we have:

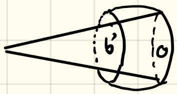
$$\mathrm{Vol}(K'_{\leq p}) = 0 \quad \& \quad \mathrm{vol}(K'_{\leq z}) \leq \mathrm{Vol}(C_{\leq z})$$
$$\forall z \text{ in the support of } C$$

So the center of gravity c^* of C is necessarily ≤ 0.

Proof:

By contradiction, we will show that it is impossible that $\exists b$ such that $\mathrm{Vol}(K'_{\leq b}) > \mathrm{Vol}(C_{\leq b})$, separating the cases $b < 0$ and $b > 0$.

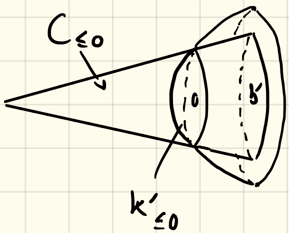
Case $b<0$: If $\mathrm{Vol}(K'_{\leq b}) > \mathrm{vol}(C_{\leq b})$, then $\exists b' \leq b < 0$ with $\mathrm{vol}(K'_{b'}) > \mathrm{Vol}(C_{b'})$. But $r(z) = \mathrm{Vol}(K'_z)^{\frac{1}{n-1}}$ is concave and $\mathrm{vol}(K'_0) = \mathrm{Vol}(C_0)$, so $\mathrm{Vol}(K'_a) \geq \mathrm{Vol}(C_a)$ $\forall b' \leq a \leq 0$



But then: $\mathrm{vol}(K'_{\leq 0}) = \mathrm{Vol}(K'_{\leq b}) + \mathrm{vol}(K'_{[b,0]})$

$$> \mathrm{vol}(C_{\leq b}) + \mathrm{Vol}(C_{[b,0]}) = \mathrm{Vol}(C_{\leq 0}) \quad ↯$$

Case $b>0$: If $\mathrm{Vol}(K'_{\leq b}) > \mathrm{vol}(C_{\leq b})$, then $\exists\, 0 < b' < b$ with $\mathrm{vol}(K'_{b'}) > \mathrm{Vol}(C_{b'})$. But $r(z) = \mathrm{vol}(K'_z)^{\frac{1}{n-1}}$ is concave & $\mathrm{vol}(K'_0) = \mathrm{Vol}(C_0)$, so $\mathrm{vol}(K'_{\leq 0}) < \mathrm{Vol}(C_{\leq 0})$ (see picture). ↯



✓ (part B)

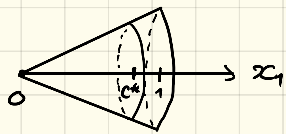
C. Let us now compute:

$$\frac{\operatorname{Vol}(K_{\le 0})}{\operatorname{Vol}(K)} = \frac{\operatorname{Vol}(K'_{\le 0})}{\operatorname{Vol}(K')} = \frac{\operatorname{Vol}(C_{\le 0})}{\operatorname{Vol}(C)} \geq \frac{\operatorname{Vol}(C_{\le c^*})}{\operatorname{Vol}(C)}$$

where c^* is the center of gravity of C (which is left to 0)

<u>Position of c^*</u>: consider a cone C of height 1 and base B:

$$\operatorname{Vol}(C) = \int_0^1 dx_1 \cdot x_1^{n-1} \cdot \operatorname{vol} B = \frac{\operatorname{vol} B}{n}$$



$$c^* = \frac{1}{\operatorname{Vol} C} \int_0^1 dx_1 \cdot x_1 \cdot x_1^{n-1} \cdot \operatorname{vol} B = \frac{\operatorname{vol} B}{(n+1) \operatorname{vol} C} = \frac{n}{n+1}$$

So $$\frac{\operatorname{Vol}(C_{\le c^*})}{\operatorname{Vol}(C)} = \left(\frac{n}{n+1}\right)^n = \frac{1}{\left(1+\frac{1}{n}\right)^n} \geq \frac{1}{e}$$ ✓ (part C)

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