Carathéodory's Theorem (Exact Version)

Let \( u \in \mathbb{R}^d \) be a point which lies in the convex hull of a set of points \( \{v_1, v_2, \ldots, v_n\} \). Thus \( u \in \text{conv}(\{v_1, \ldots, v_n\}) \).

Then \( u \) can be written as a convex combination of at most \( n+1 \) points of \( \{v_1, \ldots, v_n\} \).

That is, \( u = \sum_{i=1}^{n+1} \lambda_i v_i \), where \( \lambda_i > 0 \) and \( \sum_{i=1}^{n+1} \lambda_i = 1 \) with at least one \( \lambda_i \neq 0 \).

Note that \( n \leq m \).

The Theorem is tight. That is, why the natural question to ask is if we can use less points if we permit an \( \epsilon \)-error.
Approximate Caratheodory's Theorem

For any $p$-norm with $p > 2$, convex hull $\text{conv}(\{v_1, v_2, \ldots, v_m\}) \subseteq \mathbb{R}^n$ with $\|v_i\|_p \leq 1$ ($v_i \in B_p^n$ for $i \in \{1, \ldots, m\}$), and point $u \in \text{conv}(\{v_1, \ldots, v_m\})$ there exists $\tilde{u} \in U$ such that $\|u - \tilde{u}\|_p \leq \varepsilon$ where $\tilde{u}$ is in the convex hull of at most $\frac{4p}{\varepsilon^2}$ points of $\{v_1, v_2, \ldots, v_m\}$. That means $\tilde{u} \in \text{conv}(\{v_1, \ldots, v_{\frac{4p}{\varepsilon^2}}\}) \subseteq \text{conv}(\{v_1, \ldots, v_m\})$

Observations

1. If $\max\{\|v_i\|_p \leq 1\}$ then the # of points $\leq \frac{4p}{\varepsilon^2}$

2. $\frac{4p}{\varepsilon^2}$ is independent of the dimension $n$ of the space.

3. The original proof of the theorem is probabilistic and non-constructive.
Optimization formulation

\[ V = \begin{bmatrix} v_1 & v_2 & \cdots & v_m \end{bmatrix}, \quad x \in \Delta^m \quad \text{simplex} \]

we want to solve \( \min \| Vx - u \|_p \)

\[ \text{s.t. } x \text{ is sparse} \quad x \in \Delta^m \]

Problem: we do not know how to model the sparsity requirement.

Idea: solve the unconstrained problem iteratively using gradient descent/mirror descent and hope that the solution is sparse.

This idea will fail since the gradient update may not be sparse.

we need to formulate the problem differently.
Reformulation using Sion's minimax theorem

Remember that the dual norm of $\| \cdot \|_p$ is $\| \cdot \|_q'$ where $\frac{1}{p} + \frac{1}{q} = 1$. From the dual norm definition, we get that

$$\| x - u \|_p = \max \left\{ \langle y, x - u \rangle \mid \| y \|_q' \leq 1 \right\} = \max \left\{ \langle y, x - u \rangle \mid y \in B_q' \right\}$$

$$\Rightarrow \min_{x \in D_w} \| x - u \|_p = \min_{x \in D_w} \max_{y \in B_q'} \langle y, x - u \rangle =$$

Sion's

$$\max_{y \in B_q'} \min_{x \in D_w} \langle y, x - u \rangle =$$

$$= \max \left( -\max_{x \in D_w} \langle y, u - x \rangle \right) =$$

$$= -\min_{y \in B_q'} \max_{x \in D_w} \langle y, u - x \rangle =$$

$$= -\min_{y \in B_q'} f(y)$$
Now the unconstrained problem is equivalent to solving:

\[ \min_{y \in \mathcal{B}} f(y) \quad \text{where} \quad f(y) = \max_{x \in \Delta^u} \langle y, u - V x \rangle \]

\[ = \langle y, u - V x^y \rangle \]

\[ x^y = \arg\max_{x \in \Delta^u} \langle y, u - V x \rangle \]

dlet's calculate some (sub)gradients:

\[ x^y = \arg\max_{x \in \Delta^u} \langle y, u - V x \rangle \]

\[ \nabla f(y) = u - V x^y = -u - u_i \]

**proof**

\[ f(z) \geq f(y) + \langle u - V x^y, z - y \rangle \Rightarrow \]

\[ f(z) \geq f(y) - \langle u - V x^y, y \rangle + \langle u - V x^y, z \rangle \Rightarrow \]

\[ \max_{x \in \Delta^u} \langle z, u - V x \rangle \Rightarrow \langle z, u - V x^y \rangle \]
\[ D \varphi(x, y) = a \| y - y \|_q^2 \]

If \( f \) is \( \ell_i \)-lipschitz w.r.t. \( \| \cdot \|_p \), then with an appropriate stepsize \( \alpha \) we get

\[ \frac{1}{T} \sum \langle \nabla f(y), y + y \rangle \leq 2 \frac{D \varphi(x^{(1)})}{\sqrt{T}} y e^B \]

In the original statement we have

\[ \frac{1}{T} \sum (f(y) - f(y^*)) \]

because of the first order condition and because \( y^* \) is minimizer.

\[ \frac{1}{T} \sum \langle \nabla f(y), y + y \rangle = \frac{1}{T} \sum \langle u - \nu^{(t)}, y + y \rangle = \]

\[ \frac{1}{T} \sum (\langle u - \nu^{(t)}, y \rangle - \langle u - \nu^{(t)}, y \rangle) = \]

\[ \frac{1}{T} \sum f(y) - \frac{1}{T} \sum \langle u - \nu^{(t)}, y \rangle \geq \]

\[ \geq - \frac{1}{T} \sum \langle u - \nu^{(t)}, y \rangle, \text{ since } f(y) > 0 \]

\[ (f(y))_{\max} \leq y^* \nu^{(t)} > 0 \]

Because \( \nu = x + s \)
So, using mirror descent we can get a bound of the form

\[-\frac{1}{T} \sum \langle u - v_{(u)}, y \rangle \leq \infty \quad \forall y \in B_q\]

\[\Rightarrow \langle y, \frac{1}{T} \sum v_{(u)} - u \rangle \leq \infty \quad \forall y \in B_q\]

by definition of the dual norm

\[\| \frac{1}{T} \sum v_{(u)} - u \|_p \leq \infty\]

\[\left( \frac{1}{T} + \frac{1}{cT} = 1 \right) \quad \text{(just a reminder)}\]

Now it remains to calculate the Lipschitz parameter \( \theta \) and find a function \( \psi \) s.t. \( D_{\psi}(x, y) \geq \alpha \|x - y\|_q^2 \)

1. we have \( \| \nabla f(y) \|_p = \| u - v_{\text{const}} \|_p \leq \| u \|_p + \| v_{\text{const}} \|_p \leq 2 \) (since \( v_{\text{const}} \in B_q \) and \( u \) belongs to the convex hull of \( \{v_1, \ldots, v_m\} \)

so \( \theta = 2 \)
For $1 \leq q \leq 2$, $\psi(y) = \frac{1}{2} \|y\|_2^2$, $y \in B^q$

Why should we try the function?

Well, in the case of the classical $l_2$-norm we have $\psi(y) = \frac{1}{2} \|y\|_2^2$, $\phi(x,y) = \frac{1}{2} \|x-y\|_2^2$, and now we need a bound of the form $\phi(x,y) \geq a \|x-y\|_q^2$.

For the domain $B^q$ we have:

$$\phi(x,y) \geq \frac{q-1}{2} \|x-y\|_q^2$$

and

$$\phi(y^*,o) = \max_{y \in B^q} \phi(y,o) \leq \frac{1}{2} \quad \text{(for } q = 2)$$

So $\phi(x,y) \geq a \|x-y\|_q^2$, $a = \frac{q-1}{2} = \frac{1}{q + \frac{1}{q}}$

So to get $\varepsilon$-close we need

$$\frac{1 + \frac{1}{p}}{\frac{p - 1}{2} - 1} = \frac{p - p + 1}{2(p - 1)}$$

which is also the # of $\psi$'s that we need to use.