Lecture 10  Multiplicative Weights Update

- LWU
- how to use LWU to solve

an $\text{DP}$ \quad \min c^T x
\quad \text{s.t. } A x \geq b
\quad x \geq 0

LWU \quad \sqrt{\text{Rules of the game}}

- set of $n$ decisions/experts
- $t = 1, 2, \ldots, T$ rounds
- in each round, we select one among the $n$ decisions/experts by drawing from a probability distribution $p_t$
- let’s assume we selected decision/expert $i$, now an adversary (or the environment) reveals a loss $\ell^t_i \in [-1, 1]
\frac{\ell^t_i}{\ell^t_i}$
\text{this is equivalent to a profit}
Goal: minimize the total cost incurred by our algorithm

\( L \) we will measure the expected total cost \( \sum_{t=1}^{T} \mathbb{E}_{P_t} [C_t] \)

\( P+ \in \Delta^n, \quad C_+ \in [-1,1]^n \quad (C_t \in \Theta^m) \)

Of course, we cannot hope to get a competitive algorithm when compared to the best selection of a sequence of experts \( S_t \), even if the adversary is oblivious.

**Explanation**

At every round \( t \) of the game an adversary selects randomly an action/expert \( i \). For action/expert \( i \) \( b_i^t = -1 \), while \( \forall j \neq i \ b_j^t = 1 \). In this situation the only "reasonable" strategy is to also pick randomly one decision. Thus in each round our expected cost is
\[
\frac{n-1}{n} (1) + \frac{1}{n} (-1) = \frac{n-2}{n}
\]
while the best expert of the round has a cost of \(-1\).

Overall after \(T\) rounds, we (the algorithm) incur a cost of \(T \left(1 - \frac{2}{n}\right)\) while the best sequence of experts "incur" a cost of \(T (-1)\).

Algo cost - best sequence cost =
\[
= T \left(1 - \frac{2}{n} - (-1)\right) = T \left(2 - \frac{2}{n}\right) = O(1)
\]

What should we compare our algorithm against? Let's try with the best fixed action in hindsight!!!

\[
= T \min_{i} \sum_{t} e_i^t
\]

In order to do so we will follow a very natural idea. Update \(P_t\) increasing the probability of actions performing well in this round, and decreasing the probability of bad actions.
\[ \mu \text{ good expert } \rightarrow \text{ increase probability} \]
\[ \mu \text{ bad expert } \rightarrow \text{ decrease probability} \]

Initialization: \( \forall i \in \{1, n\}, W_i = 1 \)

for \( t = 1, 2, \ldots, T \) do

1. \( \phi^t = \frac{1}{n} \sum_{i=1}^{n} w_i^t \), \( p_i^t = \frac{w_i^t}{\phi^t} \)

(2) observe loss vector \( l^t \)

(3) update weights \( \Rightarrow w_i^{t+1} = w_i^t (1 - \mu l_i^t) \)

Theorem (performance)

Assume \( l^t \in \mathbb{B}^n \), \( \mu \leq \frac{1}{2} \). After \( T \) rounds we have

\[ \frac{1}{T} \sum_{t=1}^{T} l_i^t, p_i^t \rightleftharpoons \frac{1}{T} \sum_{t=1}^{T} l_i^t + \mu \sum_{t=1}^{T} l_i^t + \frac{\log n}{\mu} \]

for all experts \( i \)
\[
\sum_{t=1}^{T} \langle e^t, p^t \rangle \leq \sum_{t=1}^{T} e_i^t + \mu \sum_{t=1}^{T} (1 - t^\mu) + \frac{\ln n}{\mu}
\]

Assume all the costs are positive. Then the theorem simplifies to fixed expert

\[
\mu W^* \leq (1 + \mu) \text{OPT} + \frac{\ln n}{\mu}
\]

In order to be \( \mu \)-multiplicatively close to the OPT we need to pay

\[
\frac{\ln n}{\mu}
\]

**proof idea**

We will try to relate \( P^t \), which is a proxy about how experts are doing as a team vs the performance of a single expert, which intuitively is stored at \( W^t \).
Proof

\[ \Phi^{t+1} = \sum_{i=1}^{n} w_i^{t+1} = \sum_{i=1}^{n} w_i^t (1 - \mu e^t_i) = \]

\[ = \Phi^t - \mu \sum_{i=1}^{n} w_i^t e^t_i = \]

\[ = \Phi^t - \mu \cdot \Phi^t \cdot <p^t, e^t> = \]

\[ = \Phi^t (1 - \mu \cdot <p^t, e^t>) \leq e^x_{t+1} \]

\[ \leq \Phi^t \cdot e \]

\[ \Rightarrow \Phi^{t+1} = \Phi^t \cdot e - \mu \sum_{i=1}^{n} <p^t, e^t> = \]

\[ = n \cdot e \]

\[ w_i^{T+1} = w_i^T (1 - \mu e^t_i) = \prod_{t=1}^{T} (1 - \mu e^t_i) \]

\[ \geq (1 - \mu)^{\sum (e^t_i)^+} - \frac{\sum (e^t_i)^-}{\Phi^t} \]

\[ \geq (1 - \mu)^+ \cdot (1 + \mu) \]

if \( |x| > 1 \) then scale down and use \( \mu' = \min \mu \)}
since \( \Phi^{t+1} > w_i^{t+1} \) \( (w_j^{t+1} > 0 \text{ for } j \in [n]) \)

\[
=> \ln n - \mu \cdot \mu \cdot \text{w-cost} > \sum_t (e_i^t)^+ \ln(1-\mu) + \\
- \sum_t (e_i^t)^- \ln(1+\mu) =>
\]

\[
= \sum_t \ln n - \mu \cdot \mu \cdot \text{w-cost} > (\mu - \mu^2) \sum_t (e_i^t)^+ \\
\ln(1-\mu) > -\mu - \mu^2 \quad \quad (\mu - \mu^2) \sum_t (e_i^t)^- <= \\
\ln(1+\mu) \leq \mu - \mu^2
\]

from the Taylor expansion of \( \ln(1+\mu) \)
\[
\ln(1+\mu) = \mu - \mu^2/2 + \mu^3/3 - \mu^4/4 + ... \\
\text{and } \mu = 1/2
\]

\[
=> \ln n - \mu \cdot \mu \cdot \text{w-cost} > -\mu \cdot \sum (e_i^t) - \mu^2 \sum (e_i^t) \\
=> \mu \cdot \mu \cdot \text{w-cost} \leq \sum (e_i^t) + \frac{\ln n}{\mu}
\]

\( \square \)
Some observations

- The adversary is allowed to see our probability distributions

- If \( \epsilon_i \in [-\mu, \mu] \) instead of \([-1, 1]\)

we just run \( \mu \mu \mu \mu \) pretending

\[ \epsilon_i^* = \epsilon_i / \mu \in [-1, 1] \]

and get

\[ \sum_t < \rho^t, \epsilon^* > \leq \sum \epsilon_i^* + \mu \cdot \frac{\| \epsilon^* \| + \mu \mu \mu}{\mu} \]

\[ \Rightarrow \sum_t < \rho^t, \epsilon^* > \leq \sum \epsilon_i^* + \mu \cdot \frac{\| \epsilon^* \| + \mu \mu \mu}{\mu} \]
Solving LPs using UWH

using binary search the problem of solving an LP of the form
\[ \min \mathbf{c}^T \mathbf{x} \]
\[ s.t. \ A \mathbf{x} \geq \mathbf{b} \]
\[ \mathbf{x} \geq 0 \]
is equivalent to
\[ \begin{bmatrix} \mathbf{1} \ \mathbf{x} \ s.t. A \mathbf{x} \geq \mathbf{b} \end{bmatrix}, \quad \mathbf{x} \geq 0 \]

We will use UWH to solve the feasibility problem.

How to do it?

We will use UWH to maintain a probability distribution \( p^t \) which (intuitively) will inform us about which constraint is infeasible and needs a fix.

A high value \( p_i^t \) should translate to \( A_i \mathbf{x} - \mathbf{b} < 0 \) where \( \mathbf{x} \) is our current solution. To do so we associate each expert with a constraint \( A_i \mathbf{x} - \mathbf{b} > 0 \)
Let \( x \) be our current solution. Then the cost of expert \( i \) is defined as \( A_i x - b_i \). This way if constraint \( k \) is not satisfied \( (A_i x - b_i < 0) \), \( U \) will increase the probability of \( P_i \) giving us a hint to what to fix!!!

Assume now \( U \) just calculated \( P_i \). We (the DP solver) are ready to calculate a tentative solution \( x^t \). How do we do that?

\( U \) cost will be \( (P^t)^T A x - (P^t)^T b \)

Since at the end we want a feasible point we will search for \( x^t s.t. \)

\( (P^t)^T A x^t - (P^t)^T b > 0 \) \( \Rightarrow \) fixes on average in feasible constraints
Oracle assumption

We assume that given the problem

\[ \exists \mathbf{x} \geq 0 \text{ s.t. } (p)^T A x - (p)^T b \geq 0 \]

either

1. \( \text{no, it is infeasible} \)
2. \( \text{returns } x' \geq 0 \)

\( (q)^T A x' - (p)^T b > 0 \)

\( x' > 0 \)

and \( \|A_i x' - b_i\|_W \leq \delta \)

Now we have all the essential ingredients to design an algorithm and hope that it works. ☺️

(1) \( p^t \) gives us information on what to fix

(2) with the help of the Oracle we fix "very" infeasible constraints

(3) we update \( p^t \) to \( p^{t+1} \) to check how the constraints are doing with the new tentative solution \( x_t \)

(4) we hope for the best.
Initialize $p_i^{(1)} = \frac{1}{n}$ for $i \in [n]$

for $t=1$ to $T$

1) find (using the Oracle) $x_t^{(t)}$ that satisfies

$$<p^+, A x_t^t> \geq <p^+, b>$$

$$x_t^t > 0$$

$$\|A x_t^t - b_t\| \leq w$$

2) set $e_i^t = \frac{A_i x_t^t - b_t}{w}$ for $i \in [n]$

3) calculate the new probability distribution over the experts $p_i^{t+1}$ using $e_t^t$.

Return $\bar{x} = \frac{1}{T} \sum_{t=1}^{T} x_t^t$
Theorem (LWU as an dP solver)

def ε an input parameter. Then either LWU correctly concludes that the system is infeasible or it outputs $\bar{x} \geq 0 \text{ s.t. } A\bar{x} - b \geq -\varepsilon$ for $i \in [n]$

in $O\left(\frac{m^2 \log n}{\varepsilon^2}\right)$ iterations.

proof

first case: the oracle outputs that $<p^*, A\bar{x} > - <p^*, b > \geq 0$ is infeasible $\bar{x} \geq 0$

In this case, the dP is not feasible. Assume towards contradiction that $\exists \bar{x} \geq 0 \text{ s.t. } A\bar{x} > b$ then

$(p^*)^TA\bar{x} > <p^*, b >$ since $p^*_i \geq 0 \forall i$

which is a contradiction

second case: By the LWU performance theorem we have that

$\sum_{t=1}^{T} <\bar{p}^t, e_t^+ > \leq \sum_{t=1}^{T} e_t^+ + \mu \sum_{t=1}^{T} e_t^+ + \frac{\epsilon m}{\mu}$

where $e_t^+ = \frac{A_i\bar{x}-b_i}{w}$
for the RHS we have that
\[
\sum_{t=1}^{T} \langle p_t^+, e_t^+ \rangle = \frac{1}{w} \sum_{t=1}^{T} \langle p_t^+, A x_t^+ - b \rangle \geq 0
\]

because of the Oracle definition
\[
\text{(it finds } x_t^* \text{ s.t. } \langle p^+, A x^+ \rangle - \langle p^+, b \rangle \geq 0)
\]

\[
x^* \rangle 0
\]

Thus the RHS \( \geq 0 \)

\[
\Rightarrow \sum_{t} (A i x_t^+ - b_i) + \mu \sum_{t} |A i x_t^+ - b_i| + \frac{\text{w ln } n}{\mu} \geq 0
\]

\[
\Rightarrow \sum_{t} (A i x_t^+ - b_i) + \mu w T + \frac{\text{w ln } n}{\mu} \geq 0
\]

\[
\Rightarrow A i \sum_{t} x_t^+ - b_i + \mu w + \frac{\text{w ln } n}{\mu} \geq 0 \Rightarrow
\]

\[
\Rightarrow A i \bar{x} - b_i \geq -\mu w - \frac{\text{w ln } n}{\mu}
\]

\[
-\mu w = -\frac{\xi}{2} \Rightarrow \mu = \frac{\xi}{2w}
\]

\[
-\frac{\text{w ln } n}{\mu T} = -\frac{\xi}{2} \Rightarrow T = \frac{2 \text{w ln } n}{\xi} = \frac{2 \text{w ln } n}{\xi} \cdot \frac{\xi/2w}{\xi}
\]

\[
= \frac{4 \text{w}^2 \text{ln } n}{\xi^2}
\]