Lecture 10 Multiplicative Weights Update

- MWU
- how to use MWU to solve an LP

$$\min c^T x \quad \text{s.t. } Ax \geq b, \quad x \geq 0$$

MWU

Rules of the game

- set of n decisions / experts
- $t = 1, 2, \ldots, T$ rounds
- in each round, we select one among the n decisions / experts by drawing from a probability distribution P_t
- let's assume we selected decision/expert i, now an adversary (or the environment) reveals a loss

$\ell_i^t \in [-1, 1]$

t ← round

i ← decision

-1 → this is equivalent to a profit of 1

Goal: minimize the total cost incurred by our algorithm

↳ we will measure the expected total cost $\sum_{t=1}^{T} \langle p_t, \ell_t \rangle$

simplex (= set of all probability distributions over n actions) , $p_t \in \Delta^n$, $\ell_t \in [-1,1]^n$

$(\ell_t \in B_\infty^n)$

Of course we cannot hope to get a competitive algorithm when compared to the best selection of a sequence of experts!!! (even if the adversary is oblivious)

<u>explanation</u>

at every round t of the game an adversary selects randomly an action/expert i. For action/expert i $\ell_i^t = -1$, while $\forall j \neq i$ $\ell_i^t = 1$. In this situation the only "reasonable" strategy is to also pick randomly one desicion. Thus in each round our expected cost is

$\frac{n-1}{n}(1) + \frac{1}{n}(-1) = \frac{n-2}{n}$ while the best expert of the round has a cost of -1.

Overall after T rounds we (the algorithm) incurr a cost of $T\cdot\left(1-\frac{2}{n}\right)$ while the best sequence of experts "incurs" a cost of $T(-1)$.

Alg cost − best sequence cost =

$$= T\left(1-\frac{2}{n}-(-1)\right) = T\left(2-\frac{2}{n}\right) = O(T)$$

trivial

What should we compare our algorithm against? Let's try with the best fixed action in hindsight!!!

$$\Rightarrow \min_{i\in[n]} \sum_{t=1}^{T} \ell_i^t$$

In order to do so we will follow a very natural idea. Update P_t increasing the probability of actions performing well in this round, and decreasing the probability of bad actions.

MWU → good expert → increase probability
→ bad expert → decrease probability

Initialization: $\forall i \in [n]\ w_i = 1$

for $t = 1, 2, \ldots, T$ do

(1) $\Phi^{(t)} = \sum_{i=1}^{n} w_i^t$, $p_i^t = \frac{w_i^t}{\Phi^{(t)}}$

partition function

(2) observe loss vector ℓ^t

(3) update weights $\Rightarrow w_i^{t+1} = w_i^t (1 - \mu \ell_i^t)$

increase probability; decrease probability

$\ell_i^t \in [-1, 1]$

$w_i^{t+1} \in [w_i^t (1-\mu), w_i^t (1+\mu)]$

Theorem (performance)

Assume $\ell^t \in B_\infty^n\ \forall t$, $\mu \leq 1/2$. After T rounds we have

$$\sum_{t=1}^{T} \langle \ell^t, p^t \rangle \leq \sum_{t=1}^{T} \ell_i^t + \mu \sum_{t=1}^{T} |\ell_i^t| + \frac{\ln n}{\mu}$$

for all experts i

$$\sum_{t=1}^{T} \langle \ell^t, p^t \rangle \leq \sum_{t=1}^{T} \ell_i^t + \mu \sum_{t=1}^{T} |\ell_i^t| + \frac{\ln n}{\mu}$$

Assume all the costs are positive. Then the theorem simplifies to

fixed expert

$$MWU \leq (1+\mu)\,OPT + \frac{\ln n}{\mu}$$

In order to be μ-multiplicatively close to the OPT we need to pay $\ln n / \mu$.

proof idea

We will try to relate Φ^t, which is a proxy about "how experts are doing as a team vs the performance of a single expert, which intuitively is stored at w_i^t.

proof

$$\Phi^{t+1} = \sum_{i=1}^{n} w_i^{t+1} = \sum_{i=1}^{n} w_i^t (1-\mu \ell_i^t) =$$

$$w_i^t = p_i^t \cdot \Phi^t$$

$$= \Phi^t - \mu \sum_{i=1}^{n} w_i^t \ell_i^t =$$

$$= \Phi^t - \mu \cdot \Phi^t \langle p^t, \ell^t \rangle =$$

$$e^x \geq x+1$$

$$= \Phi^t (1 - \mu \langle p^t, \ell^t \rangle) \leq$$

$$\leq \Phi^t \cdot e^{-\mu \langle p^t, \ell^t \rangle}$$

$$\Rightarrow \Phi^{T+1} \leq \Phi^1 \cdot e^{-\mu \sum_{t=1}^{T} \langle p^t, \ell^t \rangle} =$$

$$= n \cdot e^{-\mu\, \mathrm{alg\_cost}}$$

$$w_i^{T+1} = w_i^T (1 - \mu \ell_i^T) = \prod_{t=1}^{T} (1 - \mu \ell_i^t)$$

$$\geq (1-\mu)^{\sum_t (\ell_i^t)^+} \cdot (1+\mu)^{-\sum_t (\ell_i^t)^-}$$

$$(1 - \mu x) \geq (1-\mu)^x, \quad x \in [0,1]$$

$$(1 - \mu x) \geq (1+\mu)^{-x}, \quad x \in [-1,0]$$

if $|x| > 1$ then scale down and use $\mu' = n \cdot \mu > \mu$

since $\Phi^{T+1} \geq w_i^{T+1}$ ($w_j^{T+1} \geq 0 \ \forall j \in [n]$)

$$\Rightarrow \ln n - \mu \cdot \text{MWU\_cost} \geq \sum_t (\ell_i^t)^+ \ln(1-\mu) + - \sum_t (\ell_i^t)^- \ln(1+\mu) \Rightarrow$$

$$\Rightarrow \ln n - \mu \cdot \text{MWU\_cost} \geq (-\mu-\mu^2) \sum_t (\ell_i^t)^+ - (\mu-\mu^2) \sum_t (\ell_i^t)^- \Rightarrow$$

$\ln(1-\mu) \geq -\mu - \mu^2$

$\ln(1+\mu) \leq \mu - \mu^2$

from the Taylor expansion of $\ln(1+\mu)$

$$\ln(1+\mu) = \mu - \mu^2/2 + \mu^3/3 - \mu^4/4 + \dots$$

and $\mu \leq 1/2$

$$\Rightarrow \ln n - \mu \, \text{MWU\_cost} \geq -\mu \cdot \sum \ell_i^t - \mu^2 \sum |\ell_i^t|$$

$$\Rightarrow \text{MWU\_cost} \leq \sum \ell_i^t + \mu \sum |\ell_i^t| + \frac{\ln n}{\mu}$$

$\blacksquare$

Some observations

- The adversary is allowed to see our probability distributions

- If $\ell_i^t \in [-w, w]$ instead of $[-1,1]$ we just run MWU pretending

$$\ell'^t_i = \ell_i^t / w \in [-1,1]$$

and get

$$\sum_t \langle p^t, \ell'^t \rangle \leq \sum \ell'^t_i + \mu \cdot \sum |\ell'^t_i| + \frac{\ln n}{\mu} \Rightarrow$$

$$\Rightarrow \sum_t \langle p^t, \ell^t \rangle \leq \sum \ell_i^t + \mu \sum |\ell_i^t| + \frac{w \ln n}{\mu}$$

Solving LPs using MWU

using Binary search the problem of solving an LP of the form $\min c^T x$

$$\text{s.t } Ax \geq b, \quad x \geq 0$$

is equivalent to $? \exists x$ s.t $Ax \geq b$, $x \geq 0$.

We will use MWU to solve the feasibility problem

How to do it?

We will use MWU to maintain a probability distribution p^t which (intuitively) will inform us about which constraint is infeasible and needs a fix.

a high value p_i^t should translate to $A_i x - b_i << 0$ where x is our current solution. To do so we associate each expert with a constraint $A_i x - b_i \geq 0$

Let x be our current solution. Then the cost of expert i is defined as $A_i x - b_i$. This way if constraint i is not satisfied ($A_i x - b_i < 0$), MWU will increase the probability of p_i^t giving us a hint to what to fix!!!

Assume now MWU just calculated p_i^t. We (the LP solver) are ready to calculate a tentative solution x^t. How do we do that?

transpose

MWU cost will be $(p^t)^T A x - (p^t)^T \cdot b$

Since at the end we want a feasible point we will search for x^t s.t

$$(p^t)^T A x^t - (p^t)^T b \geq 0$$ ⇝ fixes on average infeasible constraints

Oracle assumption

We assume that given the problem

? $\exists x$ s.t $(p)^T A \cdot x - (p)^T b \geq 0$ replies
$x \geq 0$

either ① no, it is infeasible

or

② returns x' s.t
$(p)^T A x' - (p)^T b \geq 0$
$x' \geq 0$

and $|A_i x^{(t)} - b_i| \leq w \; \forall i$

width of the oracle

Now we have all the essential ingredients to design an algorithm and hope that it works. :P

(1) p^t gives us information on what to fix
(2) with the help of the Oracle we fix "very" infeasible constraints
(3) we update p^t to p^{t+1} to check how the constraints are doing wrt the new tentative solution x_t
(4) we hope for the best.

MWU for LP solving

Initialize $p_i^{(1)} = 1/n \quad \forall i \in [n]$

for $t=1$ to T

(1) find (using the Oracle) $x^{(t)}$ that satisfies $\langle p^t, Ax^t \rangle \geq \langle p^t, b \rangle$

$x^t \geq 0$

$|A_i x^t - b_i| \leq w \quad \forall i \in [n]$

(2) set $\ell_i^t = \dfrac{A_i x^t - b_i}{w} \quad \forall i \in [n]$

(3) calculate the new probability distribution over the experts p_i^{t+1} using ℓ_i^t.

Return $\bar{x} = \dfrac{1}{T} \sum_{t=1}^{T} x^t$

Theorem (MWU as an LP solver)

Let ε an input parameter. Then either MWU correctly concludes that the system is infeasible or it outputs $\bar{x}$ s.t $A_i\bar{x} - b_i \geq -\varepsilon \ \forall i \in [n]$

$\bar{x} \geq 0$

width of the oracle

number of constraints

in $O\left(\frac{w^2 \log n}{\varepsilon^2}\right)$ iterations.

additive error in the feasibility constraints

proof

first case: the Oracle outputs that

$$\boxed{\begin{aligned} \langle p^t, Ax\rangle - \langle p^t, b\rangle \geq 0 \\ x \geq 0 \end{aligned}}$$

is infeasible

In this case the LP is not feasible.

Assume towards contradiction that $\exists\, x'$ s.t $\begin{aligned} Ax' \geq b \\ x' \geq 0 \end{aligned}$ then

$$(p^t)^T A x' \geq \langle p^t, b\rangle \quad \text{since } p_i^t \geq 0 \ \forall i$$

which is a contradiction

second case:

By the MWU performance theorem we have that

$$\sum_{t=1}^{T} \langle p^t, \ell^t\rangle \leq \sum_{t=1}^{T} \ell_i^t + \mu \sum_{t=1}^{T} |\ell_i^t| + \frac{\ln n}{\mu} \quad \forall i$$

where $\ell_i^t = \frac{A_i x^t - b_i}{w}$ $\Longrightarrow$

for the LHS we have that

$$\sum_{t=1}^{T} \langle p^t, \ell^t \rangle = \frac{1}{W} \sum_{t=1}^{T} \langle p^t, Ax^t - b \rangle \geq 0$$

because of the Oracle definition

(it finds x^t s.t. $\langle p^t, Ax^t \rangle - \langle p^t, b \rangle \geq 0$)
$x^t \geq 0$

Thus the RHS ≥ 0

$$\Rightarrow \sum_t (A_i x^t - b_i) + \mu \sum_t |A_i x^t - b_i| + \frac{w \ln n}{\mu} \geq 0$$

$$\Rightarrow \sum_t (A_i x^t - b_i) + \mu \cdot W \cdot T + \frac{w \ln n}{\mu} \geq 0$$

$1/T$

$$\Rightarrow A_i \frac{\sum x^t}{T} - b_i + \mu \cdot W + \frac{w \ln n}{\mu \cdot T} \geq 0 \Rightarrow$$

$$\Rightarrow A_i \bar{X} - b_i \geq -\mu W - \frac{w \ln n}{\mu T} \quad \forall i$$

$$-\mu W = -\frac{\varepsilon}{2} \Rightarrow \mu = \frac{\varepsilon}{2W}$$

$$-\frac{w \ln n}{\mu T} = -\frac{\varepsilon}{2} \Rightarrow T = \frac{2 w \ln n}{\varepsilon \mu} = \frac{2 w \ln n}{\varepsilon \cdot \varepsilon / 2W} =$$

$$= \frac{4 w^2 \ln n}{\varepsilon^2} \quad \blacksquare$$