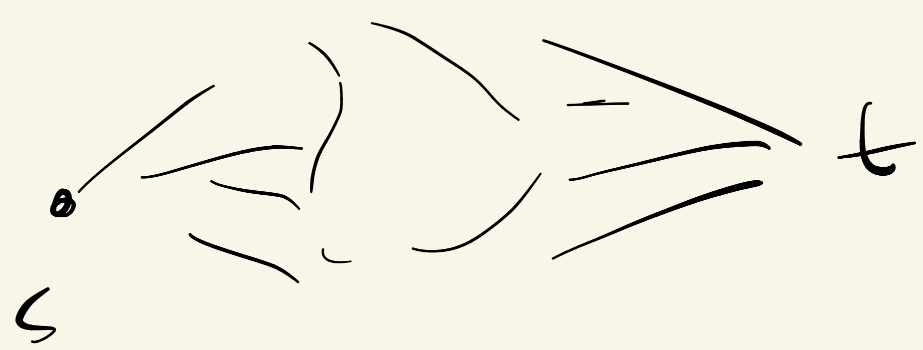
Course overview

ideas from convex geometry and convex optimization changed the way we think about algorithms

max flow problem



5 lectures → basic theorems

convex sets
separating hyperplane theorems
etc

6-17 lectures → optimization algorithms and some applications

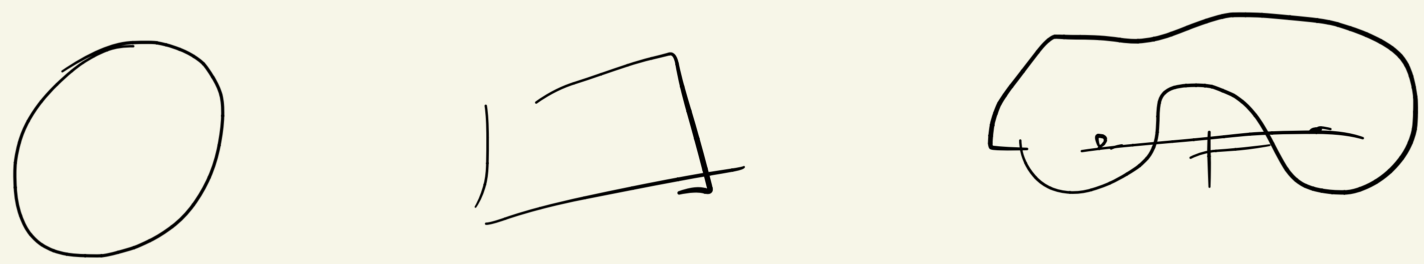
min cut
Caratheodory theorem
matroid intersection

18-24 lectures → convex geometry

Convex sets

$C \subseteq \mathbb{R}^n$ is convex iff $\forall\, x, y \in C$, $0 \le \theta \le 1$

$\theta x + (1-\theta) y$ | a set is convex if any two points that lie in the set also the line between them is contained in the set.



practical purposes enough to check $\frac{x+y}{2}$

convex combination

$\theta_1 x_1 + \dots + \theta_k x_k$ $\quad x_1, x_2, \dots, x_k$ if $\theta_i \geq 0$

and $\theta_1 + \dots + \theta_k = 1$

convex hull

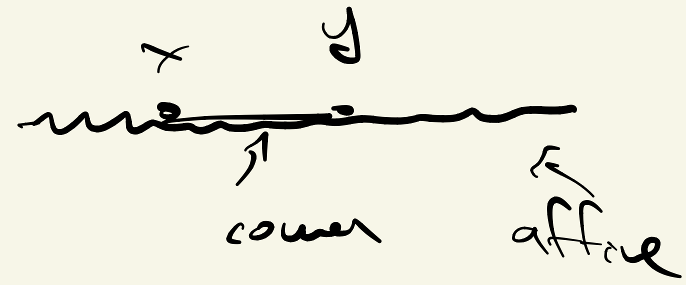
set of convex combinations

$C \to \mathrm{conv}\, C = \{\theta_1 x_1 + \dots + \theta_k x_k \mid x_i \in C, \theta_i \geq 0, 1 \leq i \leq k, \theta_1 + \dots + \theta_k = 1\}$

Affine set

A set $C \subseteq \mathbb{R}^n$ is affine if for any $x, y \in C$ and $\theta \in \mathbb{R}$ we have $\theta x + (1-\theta) y$

Difference with convex sets =



affine combination
affine hulls

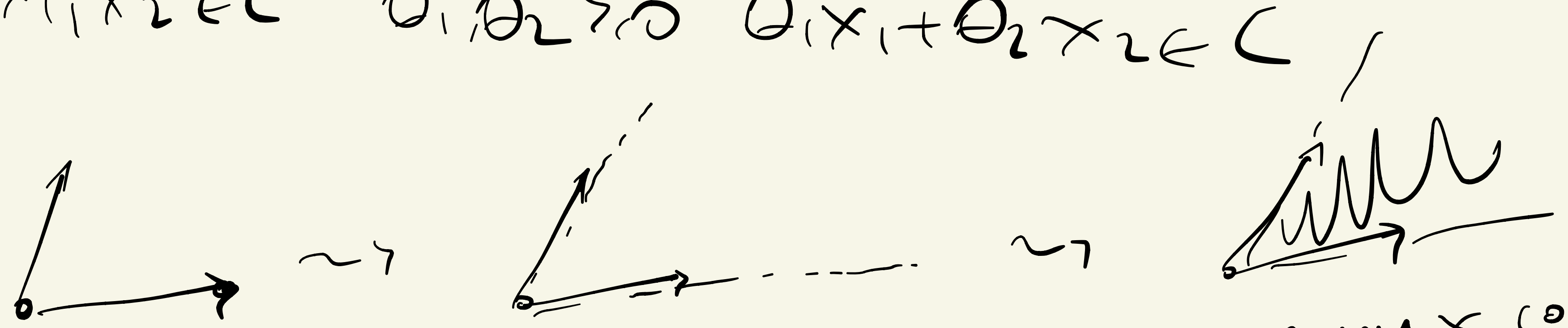
$\Rightarrow \theta_1 + \dots + \theta_k = 1$

$\theta_i \in \mathbb{R}$

Cones

$C \subseteq \mathbb{R}^n$ is a cone if $\forall x \in C, \theta \geq 0$ $\theta x \in C$

" " " convex cone if for every $x_1, x_2 \in C$ $\theta_1, \theta_2 \geq 0$ $\theta_1 x_1 + \theta_2 x_2 \in C$



Polyhedra

hyperplane $\{x \mid a^T x = b\}$

half space $\{x \mid a^T x \leq b\}$

polyhedron = intersection of finite number of

bounded polyhedron = polytope

LP → optimize a linear function over a polyhedra

Positive definite cone

$S_n = \{X \in \mathbb{R}^{n \times n} \mid X^T = X\}$ ← set of real symmetric matrices

let $M \in S_n$ then the spectral theorem for symmetric matrices tells us that

eigenvector

$$M = VDV^T = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} - & v_1 & - \\ & \vdots & \\ - & v_n & - \end{bmatrix}$$

real orthonormal

positive definite ($M \succeq 0$) (if $\lambda_i \geq 0$)

equivalent characterization

① $\lambda_i \geq 0$

② $x^T M x \geq 0 \ \forall x \in \mathbb{R}^n$

③ $M = V^T \cdot V$ for some $V \in \mathbb{R}^{n \times n}$

(positive definite $\succeq \rightsquigarrow M \succ 0$)

$S^n_+ = \{X \in S^n \mid X \succeq 0\}$ ← is a convex cone

$x^T(\theta_1 A + \theta_2 B)x \geq 0$

$S^n_{++} = \{X \in S^n \mid X \succ 0\}$

Semidefinite programming

optimize a linear function over the

positive semidefinite cone $\cap$ halfspace)

intersecting

Ellipsoids (important)

Informally: Ball $\xrightarrow[\text{transformation}]{\text{affine}}$ ellipsoid

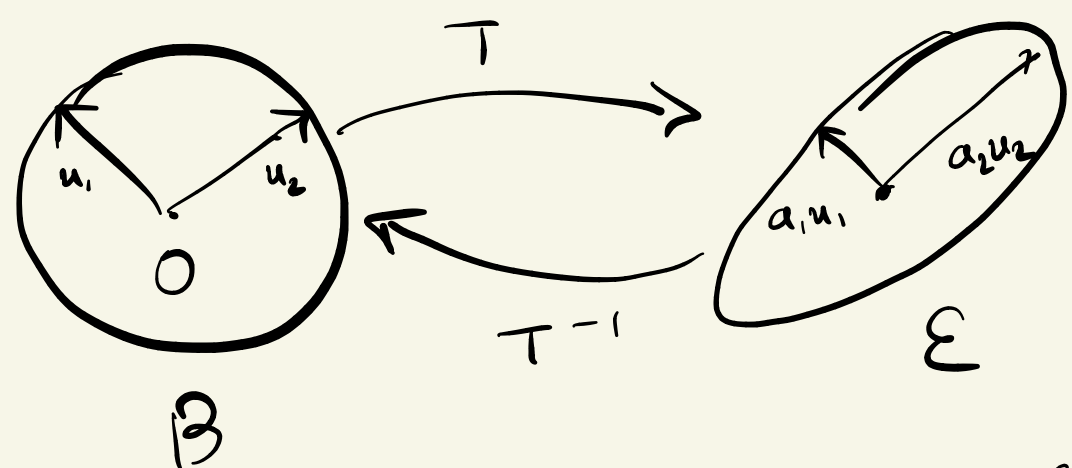
$B(x_c, r) = \{x \mid \|x - x_c\|_2 \leq r$ $\quad$ x_c center

r radius

$\|u\|_2 = \sqrt{u^T u}$

$u_1, \dots, u_n \leftarrow$ set of orthonormal vectors (semi-axes)

$a_1, a_2, \dots, a_n \geq 0$ scaling factors



$\leftarrow$ I stretched more along the u_2 direction

$x \to Tx$ $\quad$ $T = \begin{bmatrix} | & & | \\ u_1 & \dots & u_n \\ | & & | \end{bmatrix} \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{bmatrix} \begin{bmatrix} - & u_1 & - \\ & \vdots & \\ - & u_n & - \end{bmatrix}$

$a_i > 0$ $\quad$ $T = UDU^T$

$T^{-1} = (UDU^T)^{-1} = (U^T)^{-1} D^{-1} \cdot U^T = U D^{-1} U^T$

$y \in \mathcal{E} \iff T^{-1} y \in B \Rightarrow U D^{-1} U^T y \in B \Rightarrow$

$\Rightarrow \|U D^{-1} U^T y\|_2 \leq 1 \Rightarrow y^T U D^{-1} \underbrace{U^T U}_{I} D^{-1} U^T y \leq 1 \Rightarrow$

$\Rightarrow y^T \cdot U \cdot D^{-2} \cdot U^T y \leq 1$

$$y^T \cdot U \cdot D^{-2} \cdot U^T y \leq 1 \Rightarrow$$

$$\Rightarrow \mathcal{E} := \left\{ y \;\middle|\; \sum_{i=1}^{n} \frac{(y_i^T u_i)^2}{a_i^2} \leq 1 \right\}$$

$\rightarrow$ the bigger a_i the more stretch I can get in this direction

another representation

$$P \in S_{++}^n \quad (P = U D^{-2} U^T)$$

$$\mathcal{E} = \{ y \mid y^T P y \leq 1 \}$$

Note that $P \in S_{++}^n \Rightarrow P = V D' V^T$

let $P^{1/2} = V (D')^{1/2} V^T$ then $y^T P y \leq 1 \Rightarrow$

$$\Rightarrow \| P^{1/2} y \|_2 \leq 1$$

$$\mathcal{E} = \{ y \mid \| P^{1/2} y \| \leq 1 \}$$

$\rightarrow$ inverse representation you go back to the ball

Why ellipsoids?

simple enough objects to have closed form formulas for questions

like

- maximizing a linear function over an ellipsoid
- computing the volume

$$\left(\prod_{i=1}^{n} a_i\right) \cdot \mathrm{vol}(B_2^n)$$

$$\text{''}$$

$$\det(T)$$

John's theorem

rich enough to approximate reasonably "well" any convex body

Norm and norm balls

$\|\cdot\| : \mathbb{R}^n \to \mathbb{R}_+$

① $\|x\| = 0$ iff $x = 0$

② $\|t \cdot x\| = |t| \cdot \|x\| \quad \forall t \in \mathbb{R},\ x \in \mathbb{R}^n$

③ $\forall x, y \in \mathbb{R}^n \quad \|x+y\| \le \|x\| + \|y\|$

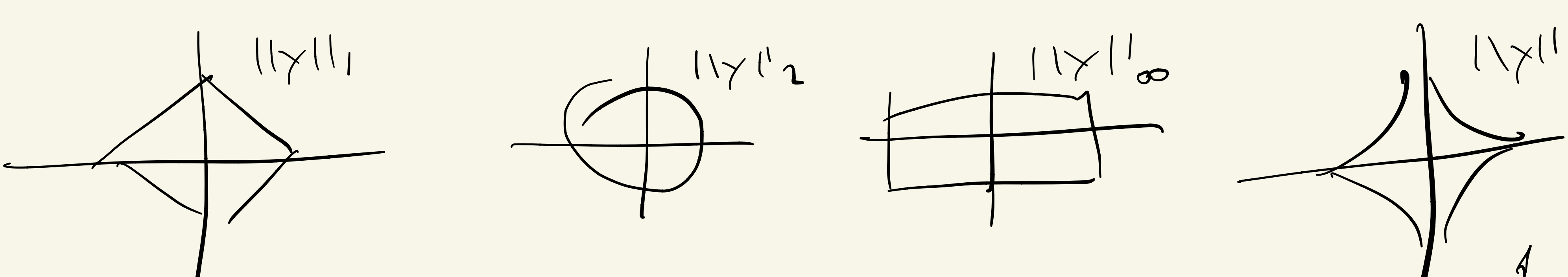
common norms

$\ell_\infty \qquad \|x\|_\infty = \max\{|x_1|, \dots, |x_n|\}$

ℓ_p, $p \ge 1 \qquad \|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$

quadratic norm $\qquad \|x\|_M = \sqrt{x^T M x} = \|M^{1/2} x\|_2$, $M \in S^n_{++}$

given a norm we can define a unit ball as $B = \{x \mid \|x\| \le 1\}$



Euclidean balls and ellipsoids just some special cases of norm balls

unit ball from ℓ_p with $p < 1$ (not a norm)

Convex functions

$f: \mathbb{R}^n \to \mathbb{R}$ is convex if $\operatorname{dom} f$ is convex and $\forall\, x, y \in \operatorname{dom} f$, $\theta \in [0,1]$

$$f(\theta x + (1-\theta) y) \le \theta f(x) + (1-\theta) f(y)$$

strictly convex $\rightsquigarrow$ $\le \;\to\; <$

concave $\rightsquigarrow$ $\le \;\to\; \ge$

epigraph connection between convex function and convex sets

$\operatorname{epi} f = \{(x,t) : x \in \operatorname{dom} f,\ t \ge f(x)\}$

lemma

$$\boxed{f \text{ is convex iff } \operatorname{epi} f \text{ is a convex set}}$$

f is convex $\Rightarrow$ $\operatorname{epi} f$ convex set

$(x_1, t_1), \dots, (x_n, t_n) \in \operatorname{epi} f$, $\lambda_1, \dots, \lambda_n \in [0,1]$ $\sum \lambda_i = 1$

$$(x,t) = \left(\sum x_i \lambda_i,\ \sum \lambda_i t_i\right) \in \operatorname{epi} f \ ?$$

$$f\left(\sum x_i \lambda_i\right) \le \sum \lambda_i f(x_i) \le \sum \lambda_i t_i \quad \checkmark$$

epi f is convex set $\Rightarrow$ f is convex

$(x, f(x)) \in \text{epi} f$ want to prove

$f(\theta x + (1-\theta) y) \leq \theta f(x) +$

$(y, f(y)) \in \text{epi} f$

$(1-\theta) f(y)$

$\Rightarrow$

$(\theta x + (1-\theta) y, \underbrace{\theta f(x) + (1-\theta) f(y)}) \in \text{epi} f$

$\geq f(\theta x + (1-\theta) y)$