1) The “Ket” and the associated Dirac or usual vector notations are:

- $|H\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\langle H | = \begin{pmatrix} 1 & 0 \end{pmatrix}$
- $|V\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\langle H | = \begin{pmatrix} 0 & 1 \end{pmatrix}$
- $\alpha |H\rangle + \beta |V\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ and $\alpha^* \langle H | + \beta^* \langle V | = \begin{pmatrix} \alpha^* & \beta^* \end{pmatrix}$

2) In Dirac notation:

\[
(\gamma^* \langle H | + \delta^* \langle V |) (\alpha |H\rangle + \beta |V\rangle) \\
= \gamma^* \alpha \langle H|H\rangle + \gamma^* \beta \langle H|V\rangle + \delta^* \alpha \langle V|H\rangle + \delta^* \beta \langle V|V\rangle \\
= \gamma^* \alpha + \delta^* \beta
\]

because $\langle H|V\rangle = \langle V|H\rangle = 0$ and $\langle H|H\rangle = \langle V|V\rangle = 1$.
The equivalent vector notation is

\[
(\gamma^* \quad \delta^*) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \gamma^* \alpha + \delta^* \beta.
\]

3) We have $R^\top = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ and $R^\dagger = R^\top^* = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$. Thus

\[
RR^\dagger = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

\[
R^\dagger R = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

Matrices satisfying $MM^\dagger = M^\dagger M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ are called unitary matrices.

Let us compute $R (\alpha |H\rangle + \beta |V\rangle)$ in Dirac notation. By linearity of matrix operations,

\[
R (\alpha |H\rangle + \beta |V\rangle) = \alpha R |H\rangle + \beta R |V\rangle \\
= \alpha i |V\rangle + \beta i |H\rangle \\
= i (\alpha |V\rangle + \beta |H\rangle).
\]
4) We have

\[ S|H\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{\sqrt{2}} (|H\rangle + i|V\rangle), \]

\[ S|V\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} (i|H\rangle + |V\rangle), \]

\[ S(\alpha |H\rangle + \beta |V\rangle) = \alpha S|H\rangle + \beta S|V\rangle \]
\[ = \frac{\alpha}{\sqrt{2}} (|H\rangle + i|V\rangle) + \frac{\beta}{\sqrt{2}} (i|H\rangle + |V\rangle) \]
\[ = \frac{\alpha + i\beta}{\sqrt{2}} |H\rangle + \frac{i\alpha + \beta}{\sqrt{2}} |V\rangle. \]

5) The semi-transparent mirror leaves photons in state \( S(\alpha |H\rangle + \beta |V\rangle) \). The state is then measured by the detector \( D \) which detects photons in state \( |V\rangle \). Therefore, the probability of finding a photon in \( D \) is the probability of finding a photon in state \( |V\rangle \) given that
photons in state $S(\alpha |H\rangle + \beta |V\rangle)$ are produced. By the measurement postulate (which will be formally introduced in Chapter 3 of the lecture note) we have

$$\text{Prob}(D) = |\langle V | S(\alpha |H\rangle + \beta |V\rangle) |^2.$$  

From the previous question we have

$$S(\alpha |H\rangle + \beta |V\rangle) = \frac{\alpha + i\beta}{\sqrt{2}} |H\rangle + \frac{i\alpha + \beta}{\sqrt{2}} |V\rangle,$$

$$\langle V | S(\alpha |H\rangle + \beta |V\rangle) = \frac{\alpha + i\beta}{\sqrt{2}} \langle V | H\rangle + \frac{i\alpha + \beta}{\sqrt{2}} \langle V | V\rangle = \frac{i\alpha + \beta}{\sqrt{2}}.$$

So we find

$$\text{Prob}(D) = \left| \frac{i\alpha + \beta}{\sqrt{2}} \right|^2 = \frac{1}{2} |i\alpha + \beta|^2 = \frac{1}{2} (\alpha^2 + \beta^2) = \frac{1}{2}.$$  

6) The state after $S$ is

$$S |H\rangle = \frac{1}{\sqrt{2}} (|H\rangle + i |V\rangle)$$

The state after $R$ is

$$RS |H\rangle = \frac{1}{\sqrt{2}} (R |H\rangle + iR |V\rangle) = \frac{i}{\sqrt{2}} (|V\rangle + i |H\rangle).$$

The state after the second $S$ is

$$SRS |H\rangle = \frac{i}{\sqrt{2}} (S |V\rangle + iS |H\rangle)$$

$$= \frac{i}{\sqrt{2}} \left( \frac{i |H\rangle + |V\rangle}{\sqrt{2}} + i \cdot \frac{|H\rangle + i |V\rangle}{\sqrt{2}} \right)$$

$$= - |H\rangle.$$  

Thus

$$\text{Prob}(D_1) = |\langle V | H\rangle |^2 = 0$$

$$\text{Prob}(D_2) = |\langle H | H\rangle |^2 = 1.$$  

All photons are detected in $D_2$! For “classical balls” we would expect a split between $D_1$ and $D_2$. For example, if $S$ act as half–half splitters we would expect $\text{Prob}(D_1) = \text{Prob}(D_2) = 1/2$. The quantum behavior is completely different!