Exercise 1  Orthonormal basis and measurement principle

Let \{\ket{x}, \ket{y}\} an orthonormal basis of \(\mathbb{C}^2\). This means that \(\braket{x}{x} = \braket{y}{y} = 1\) and \(\braket{x}{y} = 0\). Let \(\ket{\alpha} = \cos \alpha \ket{x} + \sin \alpha \ket{y}\), \(\ket{\alpha_\perp} = -\sin \alpha \ket{x} + \cos \alpha \ket{y}\), \(\ket{R} = \frac{1}{\sqrt{2}} (\ket{x} + i \ket{y})\), and \(\ket{L} = \frac{1}{\sqrt{2}} (\ket{x} - i \ket{y})\).

1) Check that \{\ket{\alpha} , \ket{\alpha_\perp}\} and \{\ket{R} , \ket{L}\} are two orthonormal basis.

2) We measure the polarization with three different measurement apparatus. The first apparatus is modeled by the basis \{\ket{x}, \ket{y}\}; the second one is modeled by the basis \{\ket{R}, \ket{L}\}; and the third one by the basis \{\ket{\alpha}, \ket{\alpha_\perp}\}. Let \(\ket{\psi} = \cos \theta \ket{x} + (\sin \theta) e^{i\varphi} \ket{y}\)

be the polarized state of a photon just before the measurement. For each of the three experiments, give the (two) possible outcoming states just after the measurement and their corresponding probabilities of outcome.

Exercise 2  Interferometer revisited

Let \(S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\), \(R = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\) be the two matrices representing a semi-transparent mirror and a perfectly reflecting mirror.

1) Compute \(S \ket{H}\), \(S \ket{V}\) and \(R \ket{H}\), \(R \ket{V}\) and give the result in Dirac notation.

2) Compute the state \(SRS \ket{H}\) as well as the probabilities \(|\braket{H}{SRS \ket{H}}|^2\) and \(|\braket{V}{SRS \ket{H}}|^2\) and verify they sum to one.

Make a picture of the experimental set-up.

3) We introduce a “dephaser” described by \(D = \begin{pmatrix} e^{i\varphi_1} & 0 \\ 0 & e^{i\varphi_2} \end{pmatrix}\) where \(\varphi_1\) and \(\varphi_2\) are two different phases (angles). Verify this matrix is unitary. Compute \(D \ket{H}\) and \(D \ket{V}\) and give the result in Dirac notation.

We consider the operation \(SRDS\). Make a picture of the experimental situation. Compute \(SRDS \ket{H}\), and the probabilities \(|\braket{H}{SRDS \ket{H}}|^2\) and \(|\braket{V}{SRDS \ket{H}}|^2\).

Verify also that the matrix \(SRDS\) is unitary and relate this fact to the other fact that the two probabilities should sum to one.