ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

Handout 35 | Principles of Digital Communications
Final exam | August 10, 2020

3 problems, 60 points
165 minutes
2 double-sided A4 sheets of notes allowed.

Good Luck!

PLEASE WRITE YOUR NAME ON EACH SHEET OF YOUR ANSWERS.

PLEASE WRITE THE SOLUTION OF EACH PROBLEM ON A SEPARATE SHEET.

PROBLEM 1. (20 points) Consider the following additive noise channel,

$$Y[i] = X[i] + \tilde{Z}[i]$$

where $X[i] \in \mathbb{R}$ is the transmitted signal and $Y[i] \in \mathbb{R}$ is the received signal. The noise $\tilde{Z}[i]$ is a colored gaussian noise, that can also be written as,

$$\tilde{Z}[i] = \sum_{j=0}^{\infty} \frac{Z[i-2j]}{2^j} = Z[i] + \frac{1}{2} \cdot Z[i-2] + \frac{1}{4} \cdot Z[i-4] + \dots$$

where $Z[i]$ are i.i.d. $\mathcal{N}(0,1)$ random variables. The transmitter uses antipodal signalling without any coding, i.e., an n bit message $b_0, \dots, b_{n-1}$ in $\{+1,-1\}^n$ is sent as $X[0], \dots, X[n-1]$ with $X[i] = b_i$, with $X[i] = 0$ if $i < 0$. We also assume that all messages are equiprobable.

(a) (4 pts) Assume that we are only given the value of $Y[0]$, determine the error probability of estimating $X[0]$ under the MAP rule.

(b) (4 pts) Determine the whitening filter for this noise, i.e., find c_j's such that, for all i

$$\sum_{j=0}^{\infty} c_j \tilde{Z}[i-j] = Z[i].$$

For the rest of the problem, define

$$\tilde{Y}[i] \triangleq \sum_{j=0}^{\infty} c_j Y[i-j].$$

(c) (4 pts) If we are only given the values of $\tilde{Y}[0], \tilde{Y}[1]$, and $\tilde{Y}[2]$, determine the MAP rule for estimating $X[2]$.

(d) (4 pts) Explain how to use the Viterbi decoder to implement the MAP decoding of the sequence $X[0], X[1], \dots$ from the observation $\tilde{Y}[0], \tilde{Y}[1], \dots$. I.e., you should find the constraint length d and a function $f(y, x_0, x_{-1}, \dots, x_{-d})$ such that performing MAP decoding is the same as finding $\{X[i]\}_{i=0}^{n-1}$ which minimizes $\sum_{i=0}^{n-1} f(\tilde{Y}[i], X[i], \dots, X[i-d])$.

(e) (4 pts) Given whitening filter output

$\tilde{Y}[0]$	$\tilde{Y}[1]$	$\tilde{Y}[2]$	$\tilde{Y}[3]$	$\tilde{Y}[4]$	$\tilde{Y}[5]$
0.4	-1	0.8	0.7	0.3	0.1,

draw the trellis diagram and find the MAP sequence $X[0], \dots, X[5]$.

PROBLEM 2. (12 points) Suppose we have a binary hypothesis testing problem where the observable Y may take k distinct values $y_1, \dots, y_k$ distributed according to a conditional probability law $p_{Y|H}$. Suppose we process Y to form a new observable

$$Z = \begin{cases} Y & \text{if } Y \in \{y_1, \dots, y_{k-1}\}, \\ y_{k-1} & \text{if } Y = y_k. \end{cases}$$

In other words the processing removes the distinction between y_{k-1} and y_k.

Let B_Y denote the Bhattacharrya bound on the error probabililty when the observation is Y and B_Z denote the Bhattacharrya bound when the observation is Z.

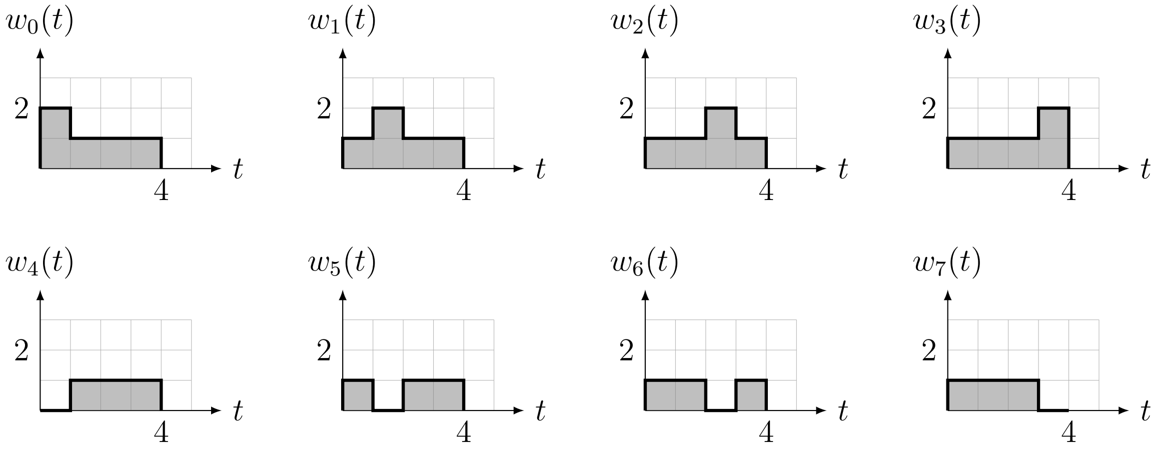
(a) (4 pts) Show that $B_Z \geq B_Y$.

Hint: You may first want to show $\sqrt{(a+b)(c+d)} \geq \sqrt{ac} + \sqrt{bd}$ for non-negative a, b, c, d.

(b) (4 pts) Find the condition for equality in (a), express the condition in terms of $\Lambda(y_{k-1})$ and $\Lambda(y_k)$ where $\Lambda(y) = p_{Y|H}(y|0)/p_{Y|H}(y|1)$ is the likelihood ratio.

(c) (4 pts) Show that when the observation Y is replaced with $\Lambda(Y)$ the Bhattacharrya bound is not changed.

PROBLEM 3. (28 points) Consider the following waveforms.



(a) (4 pts) Find an orthonormal basis for these waveforms such that the waveform elements are time shifts of a single basis function $\psi(t)$.

(b) (4 pts) Assuming equally likely messages, translate the above waveforms to form a minimum energy signal set $\tilde{\mathcal{W}} = \{\tilde{w}_i : i = 0, \dots, 7\}$. What are codewords $c_0, \dots, c_7$ that describe $\tilde{\mathcal{W}}$ in the orthonormal basis you found in (a). What is the energy per bit of $\tilde{\mathcal{W}}$?

(c) (4 pts) What can you say about the magnitude squared Fourier transforms of the waveforms $|\tilde{w}_{i,F}(f)|^2$ and $|\psi_F(f)|^2$?

(d) (4 pts) $\tilde{\mathcal{W}}$ is used to communicate over an AGWN channel with noise spectral density $N_0/2$. At the receiver, the received signal is first passed through a filter $h(t) = \mathbb{1}\{t \in [0, 1/3]\}$. How should the output of the filter be processed to ensure MAP detection?

Hint: The receiver can sample the output of filter h multiple times.

(e) (4 pts) Suppose the communication channel multiplies the transmitted waveform with either $+1$ or -1 with equal probability. We want to use the transmitter and receiver in (b) and (c) to communicate. You will notice that because of the "multiplicative" defect in the channel, certain waveforms cannot be distinguished. Describe a four element subset of $\tilde{\mathcal{W}}$ that can be distinguished despite this imperfection and how to interpret the decisions of the receiver in (c) to yield a MAP decoder.

(f) (4 pts) Find the error probability of the system in (e).

(g) (4 pts) We would like to keep the codewords as in (b) but change the function $\psi(t)$ so that the transmitted signals occupy a smaller region in the frequency domain. The partial figure below plot the magnitude squared Fourier transform of a candidate function $\tilde{\psi}(t)$ to replace $\psi(t)$. Complete the figure such that the time shifts $\{\tilde{\psi}(t-i) : i \in \mathbb{Z}\}$ forms an orthogonal basis.