SOLUTION 1.

(a) (i) The plots are shown below:

(ii) The joint density function is invariant under rotation for $\alpha = 2$ only. For this value of $\alpha$, we have $X, Y \sim \mathcal{N}(0, \frac{1}{2})$.

(b) (i) We know that we can write $(x, y)$ in polar coordinates $(r, \theta)$. Hence in general the joint distribution of $X$ and $Y$ is a function of $r$ and $\theta$. Because of circular symmetry the joint distribution should not depend on $\theta$, which means that $f_{X,Y}(x, y)$ can be written as a function of $r$. Hence if we denote this function by $\psi$ and use the independence of $X$ and $Y$, we have $f_X(x)f_Y(y) = \psi(r)$.

(ii) Taking the partial derivative with respect to $x$ and using the chain rule for differentiation, we have $f_X'(x)f_Y(y) = \psi'(r) \frac{2}{2r} = \psi'(r) \frac{r}{r}$. If we divide both sides by $xf_X(x)f_Y(y)$ we have $f_X'(x) = \frac{\psi'(r)}{r\psi(r)}$. Proceeding similarly for $y$, we obtain

$$\frac{f_X'(x)}{xf_X(x)} = \frac{\psi'(r)}{r\psi(r)} = \frac{f_Y'(y)}{yf_Y(y)}.$$
(iii) \( \frac{f'_X(x)}{xf_X(x)} \) is a function of \( x \) while \( \frac{f'_Y(y)}{yf_Y(y)} \) is a function of \( y \). Hence the only way for the equality to hold is that both of them equal a constant. If we denote this constant by \(- \frac{1}{\sigma^2}\), we reach the final result.

(iv) We have \( \frac{f'_X(x)}{f_X(x)} = -\frac{x}{\sigma^2} \). Integrating both sides we have \( \log(f_X(x)) = -\frac{x^2}{2\sigma^2} \). Hence \( f_X(x) = C \exp(\frac{-x^2}{2\sigma^2}) \) is a probability density function and so should integrate to 1, which gives \( C = \frac{1}{\sqrt{2\pi}\sigma} \). Hence \( f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp(\frac{-x^2}{2\sigma^2}) \), which shows that \( X \) and \( Y \) are Gaussian random variables.

**Solution 2.**

(a) Let \( x_E(t) = x_R(t) + jx_I(t) \). Then
\[
x(t) = \sqrt{2} \Re\{x_E(t)e^{j2\pi ft}\} \\
= \sqrt{2} \Re\{[x_R(t) + jx_I(t)]e^{j2\pi ft}\} \\
= \sqrt{2}[x_R(t) \cos(2\pi ft) - x_I(t) \sin(2\pi ft)].
\]
Hence, we have
\[
x_{EI}(t) = \sqrt{2} \Re\{x_E(t)\}
\]
and
\[
x_{EQ}(t) = \sqrt{2} \Im\{x_E(t)\}.
\]

(b) Let \( x_E(t) = \alpha(t)e^{j\beta(t)} \). Then
\[
x(t) = \sqrt{2} \Re\{x_E(t)e^{j2\pi ft}\} \\
= \sqrt{2} \Re\{\alpha(t)e^{j\beta(t)}e^{j2\pi ft}\} \\
= \sqrt{2} \Re\{\alpha(t)e^{j(2\pi ft + \beta(t))}\} \\
= \sqrt{2}\alpha(t) \cos(2\pi ft + \beta(t)).
\]
We thus have
\[
x_E(t) = \alpha(t)e^{j\beta(t)} = \frac{a(t)}{\sqrt{2}} e^{j\varphi(t)}.
\]

(c) From (b) we see that
\[
x_E(t) = \frac{A(t)}{\sqrt{2}} e^{j\varphi}.
\]
This is consistent with Example 7.9 (DSB-SC) given in the text. We can also verify:
\[
x(t) = \sqrt{2} \Re\{x_E(t)e^{j2\pi ft}\} \\
= \sqrt{2} \Re\{\frac{A(t)}{\sqrt{2}} e^{j2\pi ft}\} \\
= \Re\{A(t)e^{j(2\pi ft + \varphi)}\} \\
= A(t) \cos(2\pi ft + \varphi).
**Solution 3.**

(a) The key observation is that while $e^{j2\pi f_1 t}$ and $e^{-j2\pi f_1 t}$ are two different signals if $f_1 \neq 0$, \( \mathbb{R}\{e^{j2\pi f_1 t}\} \) and \( \mathbb{R}\{e^{-j2\pi f_1 t}\} \) are identical.

Therefore, if we fix $f_1 \neq 0$ and choose $a_1(t)$ and $a_2(t)$ so that $a_1(t)e^{j2\pi f_1 t} = e^{j2\pi f_1 t}$ and $a_2(t)e^{j2\pi f_1 t} = e^{-j2\pi f_1 t}$, we get $a_1(t) \neq a_2(t)$ and $\mathbb{R}\{a_1(t)e^{j2\pi f_1 t}\} = \mathbb{R}\{a_2(t)e^{j2\pi f_1 t}\}$.

Let $a_1(t) = e^{-j2\pi(f_c-f_1)t}$ and $a_2(t) = e^{-j2\pi(f_c+f_1)t}$. Then $a_1(t) \neq a_2(t)$ and

\[
\sqrt{2}\mathbb{R}\{a_1(t)e^{j2\pi f_1 t}\} = \sqrt{2}\mathbb{R}\{a_2(t)e^{j2\pi f_1 t}\}.
\]

(b) Let $b(t) = a(t)e^{j2\pi f_c t}$, which represents a translation of $a(t)$ in the frequency domain. If $a_F(f) = 0$ for $f < -f_c$, then $b_F(f) = 0$ for $f < 0$. Because $\mathbb{R}\{b(t)\} = \frac{1}{2}\{a(t)e^{j2\pi f_c t} + a^*(t)e^{-j2\pi f_c t}\}$, taking the real part has a scaling effect and adds a negative-frequency component. The negative spectrum is canceled by the $h_\geq$ filter, and the scaling is compensated by the $\sqrt{2}$ factors from the up-converter and down-converter. Multiplying by $e^{-j2\pi f_c t}$ translates the spectrum back to the initial position. In conclusion, we obtain $a(t)$.

(c) Take any baseband signal $u(t)$ with frequency domain support $[-f_c - \Delta, f_c + \Delta]$, $\Delta > 0$.

The signal can be real-valued or complex-valued (for example $u_F(f) = \frac{1}{2}[1_{[-f_c-\Delta, f_c+\Delta]}(f)$, which is a sinc in time domain). After we up-convert, the support of $u_F(f)$ will not extend beyond $2f_c + \Delta$. When we chop the negative frequencies we obtain a support contained in $[0, 2f_c + \Delta]$ and when we shift back to the left the support will be contained in $[-f_c, f_c + \Delta]$, which is too small to be the support of $u_F(f)$.

(d) In time domain:

\[
w(t) = \sqrt{2}\mathbb{R}\{a(t)e^{j2\pi f_c t}\},
\]

\[
a(t) \overset{a \in \mathbb{R}}{=} \frac{w(t)}{\sqrt{2}\cos(2\pi f_c t)}.
\]

In frequency domain: If $a_F(f) = 0$ for $f < -f_c$, we obtain $a(t)$ as described in (b). In the following, we consider the case $a_F(f) \neq 0$ for $f < -f_c$.

We have $w_F(f) = \frac{1}{\sqrt{2}}[a_F(f - f_c) + a_F(f + f_c)] = a_+^F(f) + a_-^F(f)$, with $a_+^F(f) = \frac{1}{\sqrt{2}}a_F(f - f_c)$ and $a_-^F(f) = \frac{1}{\sqrt{2}}a_F(f + f_c)$, respectively. These two components have overlapping support in some interval centered at 0. However, there is no overlap for sufficiently large frequencies. This means that for sufficiently large frequencies $f$ we have $w_F(f) = \frac{1}{\sqrt{2}}a_+^F(f)$, which implies that from $w_F(f)$ we can observe the right tail of $a_+^F(f)$ and use that information to remove the right tail of $a_-^F(f)$ from $w_F(f)$ (the right tails of $a_+^F(f)$ and $a_-^F(f)$ are the same because $a(t)$ is real). Hence, from $w_F(f)$ we can read more of the right tail of $a_+^F(f)$. The procedure can be repeated until we get to see $a_+^F(f)$ for all frequencies above $f_c$. At this point, using $a_F(f) = a_+^F(f + f_c)\sqrt{2}$ and the fact that $a(t)$ is real-valued, we have $a_F(f)$ for the positive frequencies, hence for all frequencies.
**Solution 4.**

\[ x(t) \sqrt{2} \cos(2\pi f_c t) = x(t) \left[ \frac{e^{j2\pi f_c t} + e^{-j2\pi f_c t}}{\sqrt{2}} \right] \]

\[ = \sqrt{2} \Re\{x_E(t) e^{j2\pi f_c t}\} \left[ \frac{e^{j2\pi f_c t} + e^{-j2\pi f_c t}}{\sqrt{2}} \right] \]

\[ = \left[ x_E(t) e^{j2\pi f_c t} + x_E^*(t) e^{-j2\pi f_c t} \right] \left[ \frac{e^{j2\pi f_c t} + e^{-j2\pi f_c t}}{\sqrt{2}} \right] \]

\[ = \frac{x_E(t) e^{j4\pi f_c t} + x_E(t) + x_E^*(t) e^{-j4\pi f_c t}}{2}. \]

At the lowpass filter output we have

\[ \frac{x_E(t) + x_E^*(t)}{2} = \Re\{x_E(t)\}. \]

The calculation for the other path is similar.

**Solution 5.**

(a) Notice that the sinusoids of \( w(t) \) have a period of \( T_s = 4 \text{ ms units of time} \), which implies that \( f_c = \frac{1}{T_s} = \frac{1}{4 \text{ms}} = 250 \text{ Hz}. \)

(b) Notice that the phase of the sinusoidal signal changes every \( T_s = 4 \text{ ms} \). (Here we have \( T_s = T_c \), but in general it is not the case. In practice we usually have \( T_s \gg T_c \). See the note at the end.)

The expression of \( w(t) \) as a function of \( t \) is:

\[ w(t) = \begin{cases} 
4 \cos(2\pi f_c t - \frac{\pi}{2}) & t \in [0, T_s[ \\
4 \cos(2\pi f_c t) & t \in ]T_s, 2T_s[ \\
4 \cos(2\pi f_c t + \pi) & t \in ]2T_s, 3T_s[ \\
4 \cos(2\pi f_c t + \frac{\pi}{2}) & t \in ]3T_s, 4T_s[ \\
\Re\left\{-4je^{j2\pi f_c t}\right\} & t \in [0, T_s[ \\
\Re\left\{4e^{j2\pi f_c t}\right\} & t \in ]T_s, 2T_s[ \\
\Re\left\{-4e^{j2\pi f_c t}\right\} & t \in ]2T_s, 3T_s[ \\
\Re\left\{4e^{j2\pi f_c t}\right\} & t \in ]3T_s, 4T_s[ \\
\Re\left\{w_E(t)e^{j2\pi f_c t}\right\} & \end{cases} \]

where

\[ w_E(t) = -\frac{4j}{\sqrt{2}} \mathbb{1}\{t \in [0, T_s]\} + \frac{4}{\sqrt{2}} \mathbb{1}\{t \in ]T_s, 2T_s[\} \]

\[ - \frac{4}{\sqrt{2}} \mathbb{1}\{t \in ]2T_s, 3T_s[\} + \frac{4j}{\sqrt{2}} \mathbb{1}\{t \in ]3T_s, 4T_s[\} \]

\[ = -j \sqrt{8T_s} \frac{1}{\sqrt{T_s}} \mathbb{1}\{t \in [0, T_s]\} + \sqrt{8T_s} \frac{1}{\sqrt{T_s}} \mathbb{1}\{t \in ]T_s, 2T_s[\} \]

\[ - \sqrt{8T_s} \frac{1}{\sqrt{T_s}} \mathbb{1}\{t \in ]2T_s, 3T_s[\} + j \sqrt{8T_s} \frac{1}{\sqrt{T_s}} \mathbb{1}\{t \in ]3T_s, 4T_s[\}. \]
If we define $\psi(t) = \frac{1}{\sqrt{T_s}}1\{t \in [0, T_s]\}$, $c_0 = -j\sqrt{8T_s}$, $c_1 = \sqrt{8T_s}$, $c_2 = -\sqrt{8T_s}$ and $c_3 = j\sqrt{8T_s}$, we get

$$w_E(t) = \sum_{i=0}^{3} c_i \psi(t - iT_s).$$

(1)

Therefore, the pulse used in the waveform former is $\psi(t) = \frac{1}{\sqrt{T_s}}1\{t \in [0, T_s]\}$, and the waveform former output signal is given by (1). The orthonormal basis that is used is $\{\psi(t - iT_s)\}_{i=0}^{3}$.

(c) The symbol sequence is $\{c_0, c_1, c_2, c_3\} = \{-j\sqrt{E_s}, \sqrt{E_s}, -\sqrt{E_s}, j\sqrt{E_s}\}$, where $E_s = 8T_s$. We can see that the symbol alphabet is $\{\sqrt{E_s}, j\sqrt{E_s}, -\sqrt{E_s}, -j\sqrt{E_s}\}$.

(d) We have:

- The output sequence of the encoder is the symbol sequence, which is $\{c_0, c_1, c_2, c_3\} = \{-j\sqrt{E_s}, \sqrt{E_s}, -\sqrt{E_s}, j\sqrt{E_s}\}$.

- The symbol alphabet contains 4 symbols. This means that each symbol represents two bits. Since the symbol rate is $f_s = \frac{1}{T_s} = 250$ symbols/s, the bit rate is $2 \times 250 = 500$ bits/s.

- The input/output mapping can be obtained by assigning two bits for each symbol in the symbol alphabet. Keeping in mind that it is better to minimize the number of bit-differences between close symbols, we obtain the following input/output mapping (which is not unique, i.e., we can obtain other mappings that satisfy the mentioned criterion): $\sqrt{E_s} \leftrightarrow 00$, $j\sqrt{E_s} \leftrightarrow 01$, $-\sqrt{E_s} \leftrightarrow 11$ and $-j\sqrt{E_s} \leftrightarrow 10$.

- Assuming that the above input/output mapping was used, we can obtain the input sequence of the encoder: 10001101.

Note that in this example, we have $T_s = T_c$, so $f_c = f_s$. This is very unusual. In practice we almost always have $f_c \gg f_s$, especially if we are using electromagnetic waves.