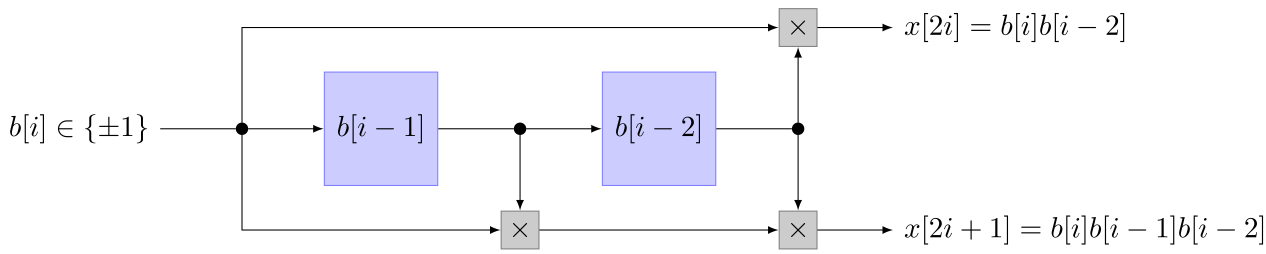
ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 31 Principles of Digital Communications

May 13, 2020. Submission Deadline : May 29, 2020. Graded Homework 3

PROBLEM 1. Consider the following convolutional code which takes k input bits and encodes it into $2k$ codeword bits.



For a given input bit sequence $\mathbf{b}$, we will refer to the corresponding codeword as $\mathbf{x}(\mathbf{b})$.

In this problem, we will study what happens when the $b[i]'s$ are not generated uniform i.i.d., but by Markov source,

$$P\big(b[i+1] = q \mid b[i] = r\big) = \begin{cases} 1-\alpha & q = r \\ \alpha & q \neq r \end{cases} \qquad \text{if } i > 0$$
$$P\big(b[0] = 1\big) = 1 - \alpha.$$

We will always assume that the convolutional code starts with memory $1, 1$.

We assume that the convolutional code is used over a BSC(δ), i.e., the channel with the following transition probability,

$$P\big(y[i] = 1 \mid x[i] = 1\big) = P\big(y[i] = -1 \mid x[i] = -1\big) = 1 - \delta$$
$$P\big(y[i] = -1 \mid x[i] = 1\big) = P\big(y[i] = 1 \mid x[i] = -1\big) = \delta$$

where $y[i]$'s are the received sequence.

a) With $\mathbf{B}$ and $\mathbf{Y}$ denoting the input bit sequence and the channel output, show that

$$P(\mathbf{B} = \mathbf{b}, \mathbf{Y} = \mathbf{y})$$

can be determined from integers f and d where $f = \sum_i \mathbb{1}\{\mathbf{b}[i] \neq \mathbf{b}[i-1]\}$ (for convenience $\mathbf{b}[-1] = 1$) is the number of transitions in the input bits and $g = \sum_i \mathbb{1}\{\mathbf{y}[i] \neq \mathbf{x}(\mathbf{b})[i]\}$ is the number of channel errors required to produce $\mathbf{y}$.

b) Using (a), describe how to assign the edge labels of the trellis diagram so that the Viterbi algorithm will perform the MAP decoding.

Due to the non-uniform prior, the bit error probability for this code depends on the transmitted sequence.

c) Consider a binary hypothesis testing problem with observation Y, show that the probability of error under MAP decoder $\hat{H}$ satisfies the following bound,

$$P(\hat{H} = 1 \mid H = -1) \leq \sqrt{\frac{P(H = 1)}{P(H = -1)}} Z$$

where Z is the Bhattacarrya parameter, i.e., $Z = \sum_y \sqrt{P_{Y|H}(y|1)P_{Y|H}(y|-1)}$.

d) Consider any input bit sequence $\mathbf{b}'$, show that the MAP decoder $\hat{\mathbf{b}}(\mathbf{y})$ fulfils

$$P\big(\hat{\mathbf{b}}(\mathbf{y}) = \mathbf{b}' \mid \forall_i b[i] = 1\big) \leq \big(2\sqrt{\delta(1-\delta)}\big)^d \sqrt{\frac{\alpha}{1-\alpha}}^{f'}$$

where $f' = \sum_i \mathbb{1}\{\mathbf{b}'[i] \neq \mathbf{b}'[i-1]\}$ (for convenience $\mathbf{b}'[-1] = 1$) and $d = \sum_i \mathbb{1}\{\mathbf{x}(\mathbf{b}')[i] \neq 1\}$.

e) We will specifically study the bit error probability of the case where $b[i] = 1$ for all $i \geq 0$. We will refer to this bit error probability as P_{e1}. Show that

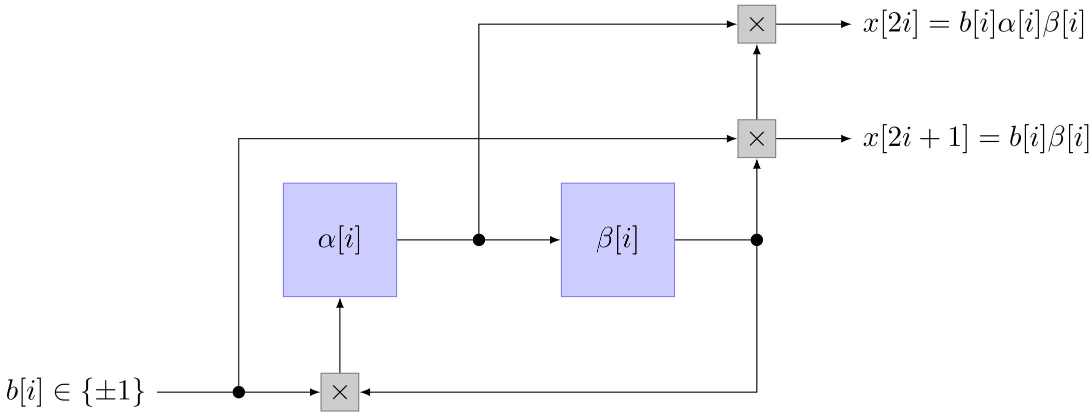
$$P_{e1} \leq \left.\frac{\partial T(I, D, F)}{\partial I}\right|_{I=1, D=2\sqrt{\delta(1-\delta)}, F=\sqrt{\alpha/(1-\alpha)}}$$

where $T(I, D, F)$ is the generating function defined as

$$T(I, D, F) = \sum_{h \in \text{Detour}} I^{i_h} D^{d_h} F^{f_h}.$$

In the generating function above, i_h is the number of bit differences between the detour's input bits and reference input bits, d_h is the number of -1's in the output sequence of the detour path, and f_h is the number of input bit transitions in the detour path.

Problem 2. Consider a "recursive" convolutional encoder that is described in the following diagram:



Formally, the system at time i is described by a state $s[i] = (\alpha[i], \beta[i]) \in \{+1, -1\}^2$ (the contents of the boxes) with $s[0] = (+,+)$ and evolution

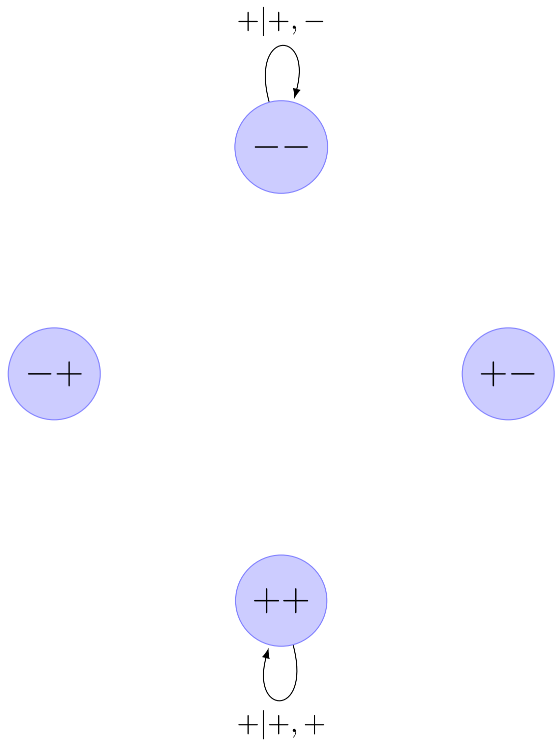
$$\alpha[i+1] = b[i]\beta[i]$$
$$\beta[i+1] = \alpha[i]$$

for $i > 0$. The encoder's output is given by

$$x[2i] = b[i]\alpha[i]\beta[i]$$
$$x[2i+1] = b[i]\beta[i].$$

Note that if "all +" bit sequence is input to the encoder, the output is the "all +" encoded sequence.

a) Complete the diagram below by drawing the edges (labeled by "$b[i] \mid x[2i], x[2i+1]$") between states (labeled by (α, β)'s).

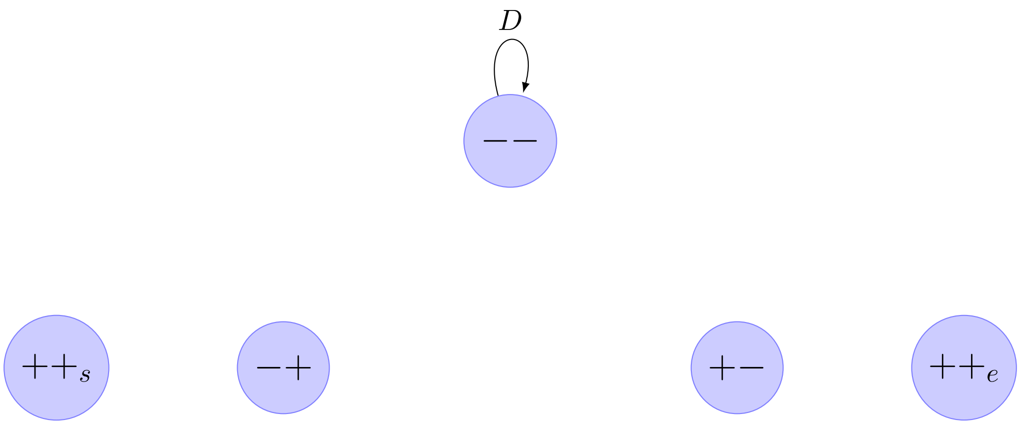


b) Draw a generic segment of the trellis diagram that corresponds to the state diagram you just found.

c) Suppose $b[0], \ldots, b[k-1]$ are k data bits. How should one choose dummy bits $b[k]$ and $b[k+1]$ to ensure that the machine returns to the initial state $++$?

d) Suppose the $x[i]$'s are sent over a channel whose output $y[i]$ is given by $x[i]+Z[i]$ where $Z[0], Z[1], \ldots$ are i.i.d. $N(0, \sigma^2)$. Describe how you would find the MAP estimate of $b[0], \ldots, b[k-1]$, from $y[0], \ldots, y[2k+3]$, assuming that $b[k]$ and $b[k+1]$ are chosen as in (c).

e) Complete the following detour graph by drawing the missing edges and labelling each edges with labels of the form "$I^i D^d$" where the power of I represents the "input distance" from the "all +" data bit sequence and the power of d represents the "output distance" from the "all +" encoded bit sequence.



f) Compute the transfer function $T(I, D)$ from $++_s$ to $++_e$.