Problem 1. Consider the following convolutional code which takes $k$ input bits and encodes it into $2k$ codeword bits.

$$(b[i], b[i - 1], b[i - 2]) \rightarrow x[2i] = b[i]b[i - 2]$$

$$(b[i], b[i - 2]) \rightarrow x[2i + 1] = b[i]b[i - 1]b[i - 2]$$

For a given input bit sequence $b$, we will refer to the corresponding codeword as $x(b)$.

In this problem, we will study what happens when the $b[i]$'s are not generated uniform i.i.d., but by Markov source,

$$P(b[i + 1] = q | b[i] = r) = \begin{cases} 1 - \alpha & q = r \\ \alpha & q \neq r \end{cases} \quad \text{if } i > 0$$

$$P(b[0] = 1) = 1 - \alpha.$$ 

We will always assume that the convolutional code starts with memory 1, 1.

We assume that the convolutional code is used over a BSC($\delta$), i.e., the channel with the following transition probability,

$$P(y[i] = 1 | x[i] = 1) = P(y[i] = -1 | x[i] = -1) = 1 - \delta$$

$$P(y[i] = -1 | x[i] = 1) = P(y[i] = 1 | x[i] = -1) = \delta$$

where $y[i]$'s are the received sequence.

a) With $B$ and $Y$ denoting the input bit sequence and the channel output, show that

$$P(B = b, Y = y)$$

can be determined from integers $f$ and $d$ where $f = \sum_i \mathbb{1}\{b[i] \neq b[i - 1]\}$ (for convenience $b[-1] = 1$) is the number of transitions in the input bits and $g = \sum_i \mathbb{1}\{y[i] \neq x(b)[i]\}$ is the number of channel errors required to produce $y$.

b) Using (a), describe how to assign the edge labels of the trellis diagram so that the Viterbi algorithm will perform the MAP decoding.

Due to the non-uniform prior, the bit error probability for this code depends on the transmitted sequence.
c) Consider a binary hypothesis testing problem with observation $Y$, show that the probability of error under MAP decoder $\hat{H}$ satisfies the following bound,

$$P(\hat{H} = 1 \mid H = -1) \leq \sqrt{\frac{P(H = 1)}{P(H = -1)}} Z$$

where $Z$ is the Bhattacharyya parameter, i.e., $Z = \sum_y \sqrt{P_{Y|H}(y|1)P_{Y|H}(y|-1)}$.

d) Consider any input bit sequence $b'$, show that the MAP decoder $\hat{b}(y)$ fulfills

$$P(\hat{b}(y) = b' \mid \forall_i b[i] = 1) \leq (2\sqrt{\delta(1-\delta)})^d \sqrt{\frac{\alpha}{1-\alpha}} f'$$

where $f' = \sum_i \mathbb{1}\{b'[i] \neq b'[i-1]\}$ (for convenience $b'[-1] = 1$) and $d = \sum_i \mathbb{1}\{x(b')[i] \neq 1\}$.

e) We will specifically study the bit error probability of the case where $b[i] = 1$ for all $i \geq 0$. We will refer to this bit error probability as $P_{e1}$. Show that

$$P_{e1} \leq \left. \frac{\partial T(I, D, F)}{\partial I} \right|_{I=1, D=2\sqrt{\delta(1-\delta)}, F=\sqrt{\alpha/(1-\alpha)}}$$

where $T(I, D, F)$ is the generating function defined as

$$T(I, D, F) = \sum_{h \in \text{Detour}} I^i_h D^{d_h} F^{f_h}.$$ 

In the generating function above, $i_h$ is the number of bit differences between the detour’s input bits and reference input bits, $d_h$ is the number of $-1$’s in the output sequence of the detour path, and $f_h$ is the number of input bit transitions in the detour path.
Problem 2. Consider a “recursive” convolutional encoder that is described in the following diagram:

Formally, the system at time $i$ is described by a state $s[i] = (\alpha[i], \beta[i]) \in \{+1, -1\}^2$ (the contents of the boxes) with $s[0] = (+, +)$ and evolution

$$\alpha[i + 1] = b[i]\beta[i]$$
$$\beta[i + 1] = \alpha[i]$$

for $i > 0$. The encoder’s output is given by

$$x[2i] = b[i]\alpha[i]\beta[i]$$
$$x[2i + 1] = b[i]\beta[i].$$

Note that if “all +” bit sequence is input to the encoder, the output is the “all +” encoded sequence.

a) Complete the diagram below by drawing the edges (labeled by “$b[i] \mid x[2i], x[2i + 1]$”) between states (labeled by $(\alpha, \beta)$’s).
b) Draw a generic segment of the trellis diagram that corresponds to the state diagram you just found.

c) Suppose $b[0], \ldots, b[k - 1]$ are $k$ data bits. How should one choose dummy bits $b[k]$ and $b[k + 1]$ to ensure that the machine returns to the initial state $++$?

d) Suppose the $x[i]$’s are sent over a channel whose output $y[i]$ is given by $x[i] + Z[i]$ where $Z[0], Z[1], \ldots$ are i.i.d. $N(0, \sigma^2)$. Describe how you would find the MAP estimate of $b[0], \ldots, b[k - 1]$, from $y[0], \ldots, y[2k + 3]$, assuming that $b[k]$ and $b[k + 1]$ are chosen as in (c).

e) Complete the following detour graph by drawing the missing edges and labelling each edges with labels of the form “$I^i D^d$” where the power of $I$ represents the “input distance” from the “all +” data bit sequence and the power of $d$ represents the “output distance” from the “all +” encoded bit sequence.

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\begin{array}{cccc}
++s & ++ & ++e \\
| & + & \| \\
- & - & - \\
\end{array}
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f) Compute the transfer function $T(I, D)$ from $++s$ to $++e$. 