SOLUTION 1.

(a) We have a binary hypothesis testing problem: The hypothesis $H$ is the answer you will select, and your decision will be based on the observation of $\hat{H}_L$ and $\hat{H}_R$. Let $H$ take value 1 if answer 1 is chosen, and value 2 if answer 2 is chosen. In this case, we can write the MAP decision rule as follows:

$$\text{Pr}\{H = 1|\hat{H}_L = 1, \hat{H}_R = 2\} \quad \frac{\hat{H}_L = 1}{\hat{H}_R = 2} \quad \text{Pr}\{H = 2|\hat{H}_L = 1, \hat{H}_R = 2\}$$

From the problem setting we know the priors $\text{Pr}\{H = 1\}$ and $\text{Pr}\{H = 2\}$; we can also determine the conditional probabilities $\text{Pr}\{\hat{H}_L = 1|H = 1\}$, $\text{Pr}\{\hat{H}_L = 1|H = 2\}$, $\text{Pr}\{\hat{H}_R = 2|H = 1\}$ and $\text{Pr}\{\hat{H}_R = 2|H = 2\}$ (we have $\text{Pr}\{\hat{H}_L = 1|H = 1\} = 0.9$ and $\text{Pr}\{\hat{H}_L = 1|H = 2\} = 0.1$). Introducing these quantities and using the Bayes rule we can formulate the MAP decision rule as

$$\text{Pr}\{\hat{H}_L = 1, \hat{H}_R = 2|H = 1\} \quad \frac{\hat{H}_L = 1}{\hat{H}_R = 2} \quad \text{Pr}\{\hat{H}_L = 1, \hat{H}_R = 2|H = 2\}$$

Now, assuming that the event $\{\hat{H}_L = 1\}$ is independent of the event $\{\hat{H}_R = 2\}$ and simplifying the expression, we obtain

$$\text{Pr}\{\hat{H}_L = 1|H = 1\} \quad \frac{\hat{H}_L = 1}{\hat{H}_R = 2} \quad \text{Pr}\{\hat{H}_R = 2|H = 1\} \quad \text{Pr}\{\hat{H}_L = 1|H = 2\} \quad \frac{\hat{H}_L = 1}{\hat{H}_R = 2} \quad \text{Pr}\{\hat{H}_R = 2|H = 2\} \quad \text{Pr}\{H = 2\},$$

which is our final decision rule.

(b) Evaluating the previous decision rule, we have

$$0.9 \times 0.3 \times 0.25 \quad \frac{\hat{H}_L = 1}{\hat{H}_R = 2} \quad 0.1 \times 0.7 \times 0.75,$$

which gives

$$0.0675 \quad \frac{\hat{H}_L = 1}{\hat{H}_R = 2} \quad 0.0525$$

This implies that the answer $\hat{H}$ is equal to 1.
Solution 2.

(a) We can write the MAP decision rule in the following way:

\[
P_{Y|H}(y|1) \begin{cases} 
\frac{\lambda_1^y e^{-\lambda_1}}{\lambda_0^y e^{-\lambda_0}} & \text{if } H = 1 \\
\frac{p_0}{1 - p_0} & \text{if } H = 0
\end{cases}
\]

Plugging in, we find

\[
\frac{\lambda_1^y e^{-\lambda_1}}{\lambda_0^y e^{-\lambda_0}} \frac{p_0}{1 - p_0},
\]

and then

\[
\left(\frac{\lambda_1}{\lambda_0}\right)^y \frac{p_0}{1 - p_0} e^{\lambda_1 - \lambda_0}
\]

Taking logarithms on both sides does not change the direction of the inequalities, therefore

\[
y \log \left(\frac{\lambda_1}{\lambda_0}\right) \frac{p_0}{1 - p_0} e^{\lambda_1 - \lambda_0}
\]

Attention: the term \(\log(\lambda_1/\lambda_0)\) can be negative, and if it is, then dividing by it involves changing the direction of the inequality.

Suppose \(\lambda_1 > \lambda_0\). Then, \(\log(\lambda_1/\lambda_0) > 0\), and the decision rule becomes

\[
y \log \left(\frac{\lambda_1}{\lambda_0}\right) \frac{p_0}{1 - p_0} e^{\lambda_1 - \lambda_0} \text{ def } \theta
\]

(b) We compute

\[
P_e(0) = \Pr\{Y > \theta|H = 0\} = \sum_{y=\lceil\theta\rceil}^{\infty} P_{Y|H}(y|0)
\]

\[
= 1 - \sum_{y=0}^{\lceil\theta\rceil} \frac{\lambda_0^y}{y!} e^{-\lambda_0},
\]

and by analogy

\[
P_e(1) = \Pr\{Y < \theta|H = 1\} = \sum_{y=0}^{\lceil\theta\rceil} P_{Y|H}(y|1)
\]

\[
= \sum_{y=0}^{\lceil\theta\rceil} \frac{\lambda_1^y}{y!} e^{-\lambda_1}
\]

Thus, the probability of error becomes

\[
P_e = p_0 \left(1 - \sum_{y=0}^{\lceil\theta\rceil} \frac{\lambda_0^y}{y!} e^{-\lambda_0}\right) + (1 - p_0) \sum_{y=0}^{\lceil\theta\rceil} \frac{\lambda_1^y}{y!} e^{-\lambda_1}
\]
Now, suppose that $\lambda_1 < \lambda_0$. Then, $\log(\lambda_1/\lambda_0) < 0$, and we have to swap the inequality sign, thus

$$y \frac{\hat{H}=0}{\hat{H}=1} \log \left( \frac{p_0}{1-p_0} e^{\lambda_1-\lambda_0} \right) \defeq \theta$$

The rest of the analysis goes along the same lines, and finally, we obtain

$$P_e = p_0 \sum_{y=0}^{\lfloor \theta \rfloor} \frac{\lambda_0^y}{y!} e^{-\lambda_0} + (1-p_0) \left( 1 - \sum_{y=0}^{\lfloor \theta \rfloor} \frac{\lambda_1^y}{y!} e^{-\lambda_1} \right)$$

The case $\lambda_0 = \lambda_1$ yields $\log(\lambda_1/\lambda_0) = 0$, so the decision rule becomes $\hat{H}=1 \gtrless \hat{H}=0 \theta$, regardless of $y$. Thus, we can exclude the case $\lambda_0 = \lambda_1$ from our discussion.

(c) Here, we are in the case $\lambda_1 > \lambda_0$, and we find $\theta \approx 4.54$. We thus evaluate

$$P_e \approx \frac{1}{3} \left( 1 - \sum_{y=0}^{4} \frac{2^y}{y!} e^{-2} \right) + \frac{2}{3} \sum_{y=0}^{4} \left( \frac{10^y}{y!} e^{-10} \right) \approx 0.03705$$

(d) We find $\theta \approx 7.5163$

$$P_e \approx \frac{1}{3} \left( 1 - \sum_{y=0}^{7} \frac{2^y}{y!} e^{-2} \right) + \frac{2}{3} \sum_{y=0}^{7} \left( \frac{20^y}{y!} e^{-20} \right) \approx 0.000885$$

The two Poisson distributions are much better separated than in (??); therefore, it becomes considerably easier to distinguish them based on one single observation $y$.

**Solution 3.** We use the Fisher–Neyman factorization theorem.

(a) Since $Y$ is an i.i.d. sequence,

$$P_{Y|H}(y|i) = \prod_{k=1}^{n} P_{Y_k|H}(y_k|i) = \lambda_{\theta i}^{\sum_{k=1}^{n} y_k} e^{-n \lambda_{\theta i}} \prod_{k=1}^{n} y_k! e^{-\lambda_{\theta i}} \frac{1}{g_i(T(y))} \prod_{k=1}^{n} (y_k)!$$

(b) Since $Z_1, \ldots, Z_n$ are i.i.d. additive noise samples,

$$f_{Y|H}(y|i) = \prod_{k=1}^{n} f_{Z_k|H}(y_k - \theta_i) = \lambda_{\theta i}^{\sum_{k=1}^{n} y_k} e^{-\lambda_{\theta i}} \sum_{k=1}^{n} y_k! \frac{1}{g_i(T(y))} \prod_{k=1}^{n} (y_k)!$$

with $h(y) = 1$. 3
**Solution 4.**

(a) It is straightforward to check that \( w_0(t) \) has unit norm, i.e., \( \|w_0(t)\| = 1 \), thus \( \psi_1(t) = w_0(t) \). With \( \psi_1(t) \) we can reproduce the first portion of \( w_1(t) \) (for \( t \) between 0 and 1). With \( \psi_2(t) \) we need to be able to describe the remaining part of \( w_1(t) \). Clearly \( \psi_2(t) \) is as illustrated below. With \( \psi_1(t) \) and \( \psi_2(t) \) we also describe the part of \( w_2(t) \) between \( t = 0 \) and \( t = 2 \). Hence \( \psi_3(t) \) is selected as the unit-norm function that matches the part of \( w_2(t) \) between \( t = 2 \) and \( t = 3 \). We immediately see that \( w_3(t) \) is also a linear combination of \( \psi_i(t) \), \( i = 1, 2, 3 \).

(b) Using the basis \( \{\psi_1(t), \psi_2(t), \psi_3(t)\} \), one can give the following representation for the waveforms \( w_i(t) \), \( i = 0, \ldots, 3 \):

\[
\begin{align*}
 w_0 &= (1, 0, 0)^T, \\
 w_1 &= (-1, 1, 0)^T, \\
 w_2 &= (1, 1, 1)^T, \\
 w_3 &= (1, 1, -1)^T
\end{align*}
\]

**Solution 5.**

(a) The optimal solution is to pass \( R(t) \) through the matched filter \( w(T - t) \) and sample the result at \( t = T \) to get a sufficient statistic denoted by \( Y \). (In this problem, \( T = 1 \).) Note that \( Y = S + N \), where \( S \) and \( N \) are random variables denoting the signal and the noise components respectively. Under \( H = i \), \( Y \sim \mathcal{N}(\alpha_i, N_0/2) \), where \( \alpha_0, \ldots, \alpha_3 \) are \( 3c, c, -c \) and \( -3c \) respectively.

Let \( \hat{X} \) be the recovered signal value at the receiver. Based on the nearest neighbor decision rule, the receiver chooses the value of \( \hat{X} \) in the following fashion:

\[
\hat{X} = \begin{cases} 
+3, & Y \in [2c, \infty) \\
+1, & Y \in [0, 2c) \\
-1, & Y \in [-2c, 0) \\
-3, & Y \in (-\infty, -2c) 
\end{cases}
\]

(b) The probability of error is given by

\[
P_e = \sum_{i=0}^{3} \frac{1}{4} \Pr\{\text{error} | H = i\} = \frac{1}{4} \left[ Q\left(\frac{c}{\sqrt{N_0/2}}\right) + 2Q\left(\frac{c}{\sqrt{N_0/2}}\right) + 2Q\left(\frac{c}{\sqrt{N_0/2}}\right) + Q\left(\frac{c}{\sqrt{N_0/2}}\right) \right] = \frac{3}{2} Q\left(\frac{c}{\sqrt{N_0/2}}\right)
\]
(c) In this case under $H = i$, $Y \sim \mathcal{N}(\alpha_i, N_0/2)$, where $\alpha_0, \ldots, \alpha_3$ are $\frac{9c}{4}$, $\frac{3c}{4}$, $-\frac{3c}{4}$ and $-\frac{9c}{4}$ respectively. Using the decision rule in (??), the probability of error is given by

$$P_e = \sum_{i=0}^{3} \frac{1}{4} \Pr \{ \text{error} | H = i \}$$

$$= \frac{1}{4} \left[ Q \left( \frac{c/4}{\sqrt{N_0/2}} \right) + Q \left( \frac{5c/4}{\sqrt{N_0/2}} \right) + Q \left( \frac{3c/4}{\sqrt{N_0/2}} \right) \right]$$

$$+ Q \left( \frac{5c/4}{\sqrt{N_0/2}} \right) + Q \left( \frac{3c/4}{\sqrt{N_0/2}} \right) + Q \left( \frac{c/4}{\sqrt{N_0/2}} \right) \right]$$

$$= \frac{1}{2} \left[ Q \left( \frac{c/4}{\sqrt{N_0/2}} \right) + Q \left( \frac{3c/4}{\sqrt{N_0/2}} \right) + Q \left( \frac{5c/4}{\sqrt{N_0/2}} \right) \right]$$

(d) The noise process $N(t)$ is a stationary Gaussian random process. So the noise component $N$ (which is the sample of match-filter output at time $T$) is a Gaussian random variable with mean

$$\mathbb{E}[N] = \mathbb{E} \left[ \int_{-\infty}^{\infty} N(t)w(t)dt \right] = \mathbb{E} \left[ \int_{0}^{1} N(t)dt \right] = 0$$

Because the process $N(t)$ is stationary, without loss of generality we choose the boundaries of the integral to be 0 and $T$ where in this problem $T = 1$.

Now, let us calculate the noise variance.

$$\text{var}(N) = \mathbb{E}[N^2] - \mathbb{E}[N]^2 = \mathbb{E}[N^2]$$

$$= \mathbb{E} \left[ \int_{-\infty}^{\infty} N(t)w(t)dt \int_{-\infty}^{\infty} N(v)w(v)dv \right]$$

$$= \mathbb{E} \left[ \int_{0}^{1} N(t)dt \int_{0}^{1} N(v)dv \right]$$

$$= \mathbb{E} \left[ \int_{0}^{1} \int_{0}^{1} N(t)N(v)dtdv \right]$$

$$= \int_{0}^{1} \int_{0}^{1} K_N(t-v)dtdv$$

$$= \int_{0}^{1} \int_{0}^{1} \frac{1}{4\alpha} e^{-|t-v|/\alpha}dtdv$$

$$= \frac{1}{2} \left( \alpha \left( e^{-1/\alpha} - 1 \right) + 1 \right)$$

Thus the new probability of error is given by

$$P_e = \sum_{i=0}^{3} \frac{1}{4} \Pr \{ \text{error} | H = i \}$$

$$= \frac{1}{4} \left[ Q \left( \frac{c}{\sqrt{\text{var}(N)}} \right) + 2Q \left( \frac{c}{\sqrt{\text{var}(N)}} \right) + 2Q \left( \frac{c}{\sqrt{\text{var}(N)}} \right) + Q \left( \frac{c}{\sqrt{\text{var}(N)}} \right) \right]$$

$$= \frac{3}{2} Q \left( \frac{c}{\sqrt{\frac{1}{2} \left( \alpha \left( e^{-1/\alpha} - 1 \right) + 1 \right)}} \right)$$