Remarks about HOSVD:

a) Core tensor $G$ is NOT di@j (diff from usual

vla"on to

b) Highly non unique (diff from usual SVD).

c) Discuss how multilinear rank approximations

important to apply for
dimensionality reduction and compression.

- Recall for Niederreiter Ecker: $\forall \theta$ The

allowed to truncated SVD $\Rightarrow$ Best Low Rank

apprrox the matrix.

- How you could hope something similar?

e.g. take truncation of $T$

$\\text{ran}_2(T) = \{ R_1, R_2, R_3 \}$

\[ \tilde{T} = \sum_{p=1}^{k_1} \sum_{q=1}^{k_2} \sum_{r=1}^{k_3} G_{pqr} \tilde{u}_p \otimes \tilde{v}_q \otimes \tilde{w}_r \]

$k_1 < R_1; k_2 < R_2; k_3 < R_3$
One can prove that:

\[ \| T - \tilde{T} \|_F^2 \leq \| T - \tilde{T} \|_F^2 \]

where

\[ \tilde{T} = \arg \min_{\| \tilde{T} \|_F^2} \| T - \tilde{T} \|_F^2 \]

\[ \text{rank}(\tilde{T}) = \{k_1, k_2, k_3\} \]

- The Naive truncation of a core tensor $G$ does not give the Best Multilinear Approx of $T$.
- However, $\tilde{T}$ exists! However, that leave systematic algorithms to compute it.

- So all in all, the Tucker or HOSVD gives a pretty good approx of $T$.
4) Last Remark.

Does a truncation of the residual tensor decay fire a Good Oppress of a tensor in any sense?

\[ R \]

\[ T = \sum_{r=1}^{R} \alpha_r \otimes b_r \otimes c_r, \quad \text{rank} \ (T) = R. \]

\[ \therefore \text{Minimize } \| T - T' \|_2 \text{ over tensors } T' \text{ with } \text{rank} \ (T') = K < R. \]

\[ \rightarrow \text{This problem is not well defined because the minimum is not obtained in span of tensors of rank } \text{span} \ (T') = K. \]

Recall contain: \[ W = e_0 \otimes e_1 \otimes e_2 + e_0 \otimes e_0 \otimes e_1, \]

\[ e_0 = (0), \quad e_1 = (1), \quad e_2 = (0). \]

\[ \therefore \text{rank} \ (W) = 3. \]

\[ \text{Take } \exists D_e, \text{ rank} \ (D_e) = 2 \text{ s.t.} \]

\[ \lim_{\varepsilon \to 0} D_e = W. \]

\[ \text{Jump of rank} \]
Prove the theorem of Tucker.  

This will give us an algorithm for HOSVD.

\[ T = \begin{pmatrix} T_1 & \cdots & T_p \end{pmatrix}, \quad \text{where} \quad T_{c1}, T_{c2}, T_{c3} \text{ are found easily from } T_{c3} \]  

Each Padriciwha has an (matrix) SVD:

1. \( T_{c1} = U_{c1}^{(1)} \Sigma_{c1}^{(1)} V_{c1}^{(1) T} \) \( I_1 \times R_1 \times R_1 \times I_2 \times I_3 \)  
   \( R_1 \times R_1 \times R_1 \times I_2 \times I_3 \)  
   \( R_1 \times R_1 \times R_1 \times I_2 \times I_3 \)  
   \( R_1 \times R_1 \times R_1 \times I_2 \times I_3 \)  
   \( R_1 \times R_1 \times R_1 \times I_2 \times I_3 \)  

   because \( \text{ran} (T) = \{ R_1, R_2, R_3 \} \).  
   \( \text{rank} (T_{c1}) \)  

2. \( T_{c2} = U_{c2}^{(2)} \Sigma_{c2}^{(2)} V_{c2}^{(2) T} \) \( I_2 \times R_2 \times R_2 \times I_1 \times I_3 \)  
   \( R_2 \times R_2 \times R_2 \times I_1 \times I_3 \)  
   \( R_2 \times R_2 \times R_2 \times I_1 \times I_3 \)  
   \( R_2 \times R_2 \times R_2 \times I_1 \times I_3 \)  
   \( R_2 \times R_2 \times R_2 \times I_1 \times I_3 \)  

3. \( T_{c3} = U_{c3}^{(3)} \Sigma_{c3}^{(3)} V_{c3}^{(3) T} \) \( I_3 \times R_3 \times R_3 \times I_1 \times I_2 \)  
   \( R_3 \times R_3 \times R_3 \times I_1 \times I_2 \)  
   \( R_3 \times R_3 \times R_3 \times I_1 \times I_2 \)  
   \( R_3 \times R_3 \times R_3 \times I_1 \times I_2 \)  
   \( R_3 \times R_3 \times R_3 \times I_1 \times I_2 \)
The idea is to consider the following multilinear form of $T$:

\[
\begin{bmatrix}
\frac{1}{n!} (U^{(1)}, U^{(2)}, U^{(3)})
\end{bmatrix}^p q r \quad R \times I,
\]

\[
= \sum_{p=1}^{R_1} \sum_{q=1}^{R_2} \sum_{r=1}^{R_3} \frac{1}{p! q! r!} (U^{(1)})^p (U^{(2)})^q (U^{(3)})^r \quad P \times Q \times R
\]

\[
= G_{pqr} \quad 1 \leq p \leq R_1, \quad 1 \leq q \leq R_2, \quad 1 \leq r \leq R_3.
\]

What remains to be done is to check that

\[
\sum_{p=1}^{R_1} \sum_{q=1}^{R_2} \sum_{r=1}^{R_3} G_{pqr} U^{(1)} U^{(2)} U^{(3)} = T
\]

which is equal to the original term $T$. You recognize here

\[
\sum_{p,q,r=1}^{R_1, R_2, R_3} G_{pqr} \mu^p \otimes \mu^q \otimes \mu^r = T.
\]
This alg is just a linearly related.

Conclusion: summarize Teker algo T HOSVD.

1. From \( T \) look at \( T_{(1)} \), \( T_{(2)} \), \( T_{(3)} \) achieve one.

2. SVD for \( T_{(1)} \), \( T_{(2)} \), \( T_{(3)} \)

\( R_1 \), nonzero singular values

\( R_2 \), non-zero singular values

\( [\tilde{u}_1 \ldots \tilde{u}_{R_1}] = U \)

\( [\tilde{v}_1 \ldots \tilde{v}_{R_2}] = V \)

\( [\tilde{w}_1 \ldots \tilde{w}_{R_3}] = W \)

\( T_{(1)} = U \Sigma V^T \quad T_{(2)} = V \Sigma^T V \)

\( \Sigma \) non-inverted

3. Compute the core tensor

\[ G_{psr} = \sum_{\alpha \beta \gamma}^T \alpha \beta \gamma \]

4. Finally, you have the decomps:

\[ \frac{T}{\alpha \beta \gamma} = \sum_{p,q,r} G_{psr} \tilde{u}_p \otimes \tilde{v}_q \otimes \tilde{w}_r. \]