**Tensor Decompositions & Tensor Rank**

Goal: Thus for order-3 tensors important primitive for studying also higher order tensors.

**Definition of Tensor Rank**

**Rank-one tensors (elementary bricks)**

- Take three vectors $a \in \mathbb{R}^{I_1}$, $b \in \mathbb{R}^{I_2}$, $c \in \mathbb{R}^{I_3}$

\[ a^\alpha, \alpha = 1 \ldots I_1, \quad b^\beta, \beta = 1 \ldots I_2, \quad \gamma = 1 \ldots I_3 \]

- Form the tensor product (or outer product)

\[ a \otimes b \otimes c = T \]

\[ T^{\alpha \beta \gamma} = a^\alpha b^\beta c^\gamma \]

3-dim array of $I_1 I_2 I_3$ numbers.
Rank - R tensor. \( R \geq 1 \).

is a multi-ery (3-arrays) that can be written as a sum of \( R \) rank-one tensors with \( R \) being the minimal possible number of terms.

\[
T = \sum_{i=1}^{R} a_i \otimes b_i \otimes c_i
\]

\( R \) should be the smallest possible to qualify for Tensor-Rank.

Terminology: such decompositions are also called "tensor factorizations" or "polyadic decompositions".

Picture:
Adopt the following notation:

\[ T = \sum_{i=1}^{2} a_i \otimes b_i \otimes c_i \]

\[ A = \begin{bmatrix} a_1 \cdots a_2 \end{bmatrix} \leftarrow I_1 \times \mathbb{R} \]

\[ \text{column vectors } a_i = \begin{bmatrix} a^{(i)} \end{bmatrix}_{\alpha=1}^{\alpha=I_1} \]

\[ B = \begin{bmatrix} b_1 \cdots b_2 \end{bmatrix} \leftarrow I_2 \times \mathbb{R} \]

\[ C = \begin{bmatrix} c_1 \cdots c_2 \end{bmatrix} \leftarrow I_3 \times \mathbb{R} \]
Remarks about the notion of tensor rank:

1) For metricien "tensor - rank" \( \Leftrightarrow \) usual lin-algebra notion of rank.

\[ \dim (\text{row span}) = \dim (\text{column span}). \]

For tensors "Tensor - Rank" is not a linear- alg (of order \( p \geq 3 \)) notion.

Moreover, there are other notions of Rank (see later lects).

2) Non trivial to decompose a tensor in a minimal \# of rank one terms.

- Unicity?
- Algorithms?

Rule of thumb: \( R \) is "small" w.r.t. to \( I_1, I_2, I_3 \)

\( \rightarrow \) decomp tends to be unique \( \rightarrow \) efficient algms are known.

\( R \) is "moderate"

\( \rightarrow \) decomp is unique \( \rightarrow \) eff algms are not always known.

\( R \) is "large" \( \rightarrow \) dont even have unicity.
Basic Theorem [about order 3-tensors, 1970’s]

Jenrich, Carroll, Hansman ...

- Let $A = [a_1, \ldots, a_r]$ $I_1 \times R$
- $B = [b_1, \ldots, b_r]$ $I_2 \times R$
- $C = [c_1, \ldots, c_r]$ $I_3 \times R$

- $A$ & $B$ have full column rank $R$. (in particular $R \leq \min(I_1, I_2)$)
- $C$ has pairwise indep columns: $\forall i, j. \quad c_i \neq b_j c_j$
- Take a 3-tensor $T$

$$T = \sum_{i=1}^{r} a_i \otimes b_i \otimes c_i$$

- Then the tensor (or multilinear form) $T_{x y z}$ can be decomposed in a unique way in a sum of at most 12 rank-one terms. And the tensor rank of $T$ is $R$.

- Qualification for uniqueness: up to trivial rescaling of $\alpha, \beta, \gamma \in \mathbb{R}$ such that $\alpha \gamma = \beta \gamma$.

\[ a_i \otimes b_i \otimes c_i \]
Remarks:

1) The theorem will be proved in a constructive way and the proof will thus give also an algorithm poly(I, I_2, I_3).

2) For this theorem we have R ≤ min(I_1, I_2) in this sense R is "small" - a non-unique & algorithmic result -

3) For R > min(I_1, I_2), len is known.

As an aside state the following result:

- Suppose that K_a, K_b, K_c are Kruskal ranks of A, B, C.

(K_a is the KR if it is the max integer such that all subsets of K_a ranks in I_0, ..., I_R are lin-indep; I_f K_a+1 subset that is lin-dep)

if 2R+2 ≤ K_a + K_b + K_c in decay of T Then this decay is unique.

- For example I_1 = I_2 = I_3 = N & R > N imagine K_a = K_b = K_c = N + R ≥ 3N - 2.

here you have non-uniqueness result for "moderately rank" N < R < \frac{3N-2}{2}.

BUT NO EFFICIENT ALGO'S ARE KNOWN!
4) This is "surprising" in the sense that for $2$-tensors (metrics) it does not hold.

Recall the "relation problem":

\[ A = \begin{bmatrix} a_1 & a_2 \end{bmatrix} \] \hspace{1cm} R = 2.

\[ B = \begin{bmatrix} b_1 & b_2 \end{bmatrix} \] \hspace{1cm} R = 2.

$A$ & $B$ have rank $R = 2$.

\[ T = a_1 \otimes b_1 + a_2 \otimes b_2 \]

This decomposition is not unique! Why?

\[ T = a_1 b_1^T + a_2 b_2^T \]

\[ = \begin{bmatrix} a_1, a_2 \end{bmatrix} \begin{bmatrix} b_1^T \\ b_2^T \end{bmatrix} \]

\[ = A B^T \]

\[ = A R R^T B^T = (AR) (BR)^T = A B^T \]

Break Next Video Proof The

\[ = a_1 \otimes b_1 + a_2 \otimes b_2 \]