ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout - Principles of Digital Communications

Quiz 1 -

3 problems.
30 minutes.
No notes allowed.
Good Luck!

Name/Surname :

Grade :

PROBLEM 1. Consider the following hypothesis testing problem:

$$\begin{aligned} H_0 &: f_{Y|H}(y|0) = \exp(-y) \\ H_1 &: f_{Y|H}(y|1) = 2\exp(-2y) \\ H_2 &: f_{Y|H}(y|2) = 2y\exp(-y^2), \end{aligned}$$

where $y \geq 0$. We want to decide whether $H = 0$ or $H \neq 0$. To this end, we will design an estimator

$$\hat{H}_\alpha(y) = \begin{cases} 0 & y \geq \alpha \\ 1 & y < \alpha. \end{cases}$$

The estimator is evaluated using the following metrics:

$$\begin{aligned} p_{\text{det}} &:= P_{\hat{H}_\alpha(Y)|H}(0|0) \\ p_{\text{fp}} &:= \max\{P_{\hat{H}_\alpha(Y)|H}(0|1), P_{\hat{H}_\alpha(Y)|H}(0|2)\}. \end{aligned}$$

A good estimator will have high probability of detection p_{det} and low probability of false positive p_{fp}.

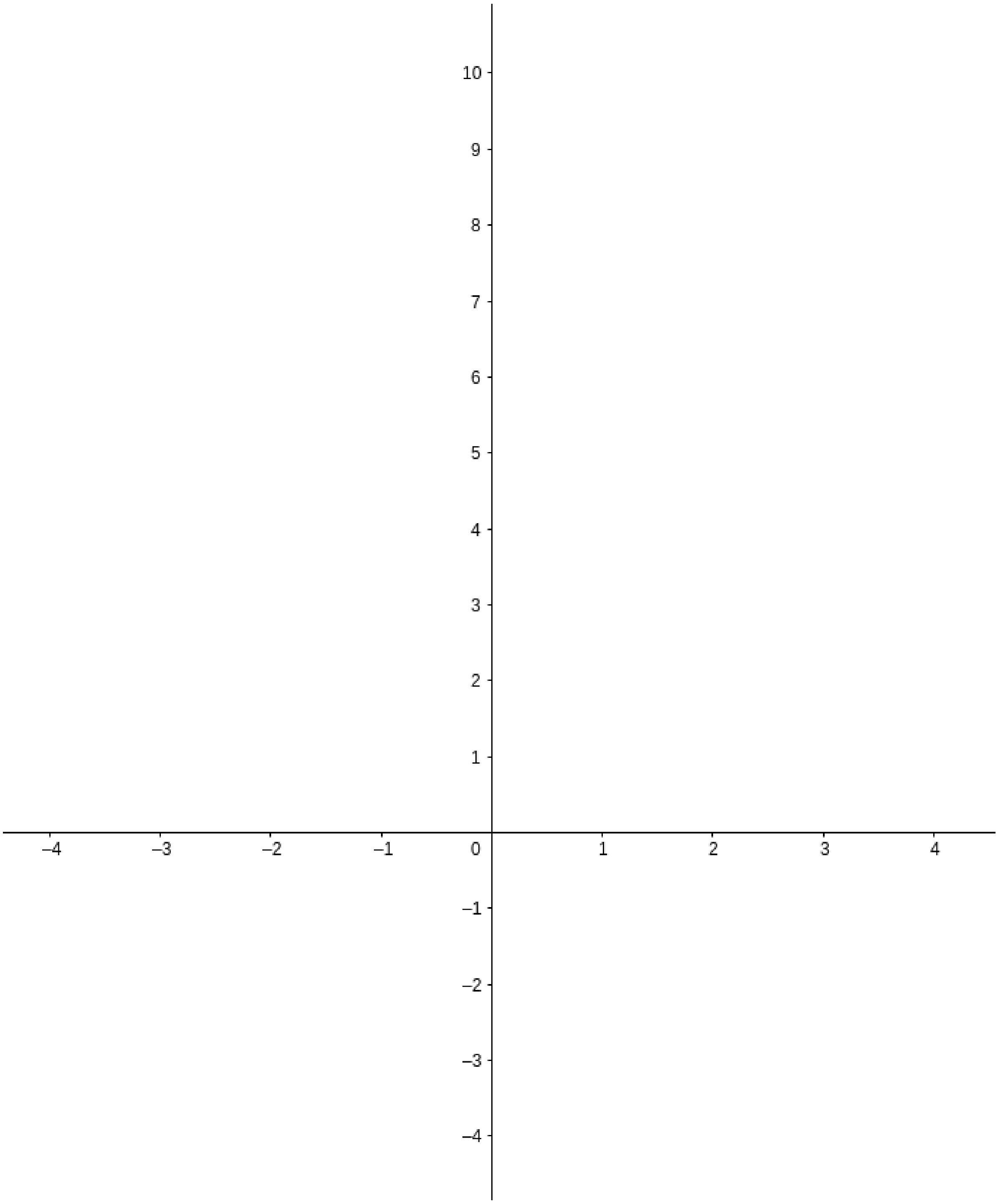
a. Calculate the following probabilities : $P_{\hat{H}_\alpha(Y)|H}(0|0)$, $P_{\hat{H}_\alpha(Y)|H}(0|1)$, and $P_{\hat{H}_\alpha(Y)|H}(0|2)$. [Hint : $\frac{d}{dx}\exp(\lambda x) = \lambda\exp(\lambda x)$ and $\frac{d}{dx}\exp(\lambda x^2) = 2\lambda x\exp(\lambda x^2)$.]

$P_{\hat{H}_\alpha(Y)|H}(0|0) =$

$P_{\hat{H}_\alpha(Y)|H}(0|1) =$

$P_{\hat{H}_\alpha(Y)|H}(0|2) =$

b. Sketch a plot of all points $(-\ln p_{\text{det}}, -\ln p_{\text{fp}})$, $0 \leq -\ln p_{\text{det}} \leq 3$, that can be achieved using $\hat{H}_\alpha$ when we vary α.



Problem 2. Consider a binary hypothesis test with observation $Y = \begin{bmatrix} Y_1 & Y_2 & Y_3 & Y_4 \end{bmatrix}^T$. Under hypothesis $H = i$, the observation is given by $Y = (-1)^i \mu + Z$ where

$$Z = \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \end{bmatrix} \sim \mathcal{N}\left(0, \sigma^2 \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \right)$$

and $\mu = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T$.

For each of the following functions, indicate if it is a sufficient statistic, and briefly explain why. [Hint : Do not apply Fisher-Neyman factorization. Observe instead that $\mathbb{E}[(Z_1 + Z_4)^2] = \mathbb{E}[(Z_2 + Z_3)^2] = 0$]

a. $T_1(Y) = Y_1 + Y_2 + Y_3 + Y_4$

b. $T_2(Y) = Y_1 + Y_2 - \frac{Y_3 + Y_4}{2}$

c. $T_3(Y) = Y_1 - Y_4 + \frac{Y_2 - Y_3}{2}$

d. $T_4(Y) = \langle \begin{bmatrix} p & 1-p & -p & p-1 \end{bmatrix}, Y \rangle$ for some $p \in \mathbb{R}$

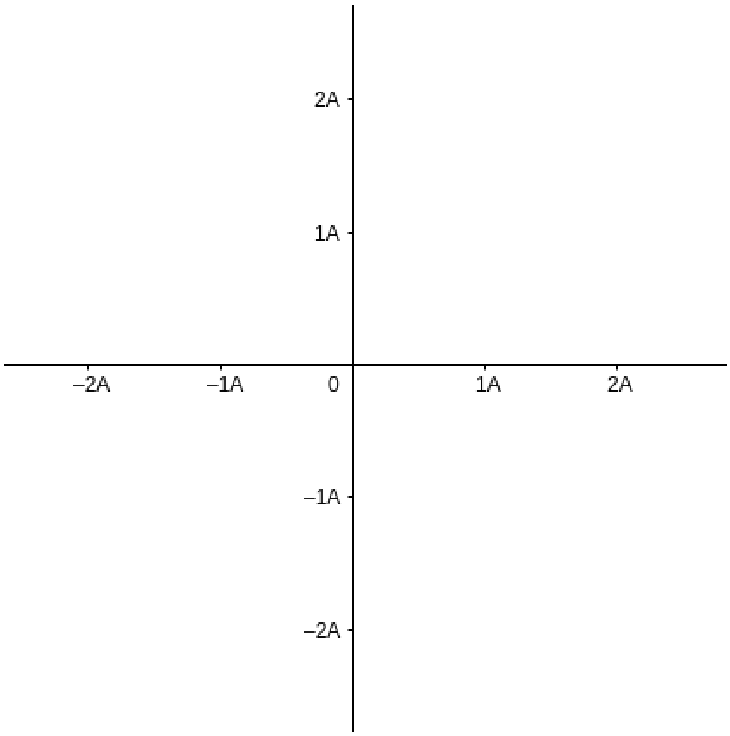
Problem 3. Assume that $H \in \{0,1,2,3,4\}$. For each $H = i$, the transmitter transmits codeword μ_i and the receiver observes Y where

$$Y = \mu_i + Z \qquad\qquad Z \sim \mathcal{N}\left(0, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right).$$

The codewords are:

$$\mu_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad -\mu_1 = \mu_2 = \begin{bmatrix} 2A \\ 0 \end{bmatrix} \qquad -\mu_3 = \mu_4 = \begin{bmatrix} 0 \\ 2A \end{bmatrix}.$$

a. Sketch the decision regions assuming that all messages $P_H(i)$ are equally likely.



b. Calculate $P(Error|H = 0)$ for the decision regions you found in part (a).