Handout - Principles of Digital Communications
Quiz 1

3 problems.
30 minutes.
No notes allowed.
Good Luck!

Name/Surname :

Grade :
**Problem 1.** Consider the following hypothesis testing problem:

\[ H_0 : f_{Y|H}(y|0) = \exp(-y) \]
\[ H_1 : f_{Y|H}(y|1) = 2 \exp(-2y) \]
\[ H_2 : f_{Y|H}(y|2) = 2y \exp(-y^2), \]

where \( y \geq 0 \). We want to decide whether \( H = 0 \) or \( H \neq 0 \). To this end, we will design an estimator

\[ \hat{H}_\alpha(y) = \begin{cases} 
0 & y \geq \alpha \\
1 & y < \alpha.
\end{cases} \]

The estimator is evaluated using the following metrics:

\[ p_{det} := P_{\hat{H}_\alpha(Y)|H}(0|0) \]
\[ p_{fp} := \max\{P_{\hat{H}_\alpha(Y)|H}(0|1), P_{\hat{H}_\alpha(Y)|H}(0|2)\}. \]

A good estimator will have high probability of detection \( p_{det} \) and low probability of false positive \( p_{fp} \).

a. Calculate the following probabilities: \( P_{\hat{H}_\alpha(Y)|H}(0|0) \), \( P_{\hat{H}_\alpha(Y)|H}(0|1) \), and \( P_{\hat{H}_\alpha(Y)|H}(0|2) \).

[Hint: \( \frac{d}{dx} \exp(\lambda x) = \lambda \exp(\lambda x) \) and \( \frac{d}{dx} \exp(\lambda x^2) = 2\lambda x \exp(\lambda x^2) \).]

\[ P_{\hat{H}_\alpha(Y)|H}(0|0) = \]

\[ P_{\hat{H}_\alpha(Y)|H}(0|1) = \]

\[ P_{\hat{H}_\alpha(Y)|H}(0|2) = \]
b. Sketch a plot of all points \((-\ln p_{\text{det}}, -\ln p_{\text{fp}}), 0 \leq -\ln p_{\text{det}} \leq 3\), that can be achieved using \(\hat{H}_\alpha\) when we vary \(\alpha\).
Problem 2. Consider a binary hypothesis test with observation \( Y = [Y_1 \ Y_2 \ Y_3 \ Y_4]^T \). Under hypothesis \( H = i \), the observation is given by \( Y = (-1)^i \mu + Z \) where

\[
Z = \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \end{bmatrix} \sim \mathcal{N} \left( \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \sigma^2 \right)
\]

and \( \mu = [1 \ 1 \ 1 \ 1]^T \).

For each of the following functions, indicate if it is a sufficient statistic, and briefly explain why. [Hint: Do not apply Fisher-Neyman factorization. Observe instead that \( \mathbb{E}[(Z_1 + Z_4)^2] = \mathbb{E}[(Z_2 + Z_3)^2] = 0 \)]

a. \( T_1(Y) = Y_1 + Y_2 + Y_3 + Y_4 \)

b. \( T_2(Y) = Y_1 + Y_2 - \frac{Y_3 + Y_4}{2} \)

c. \( T_3(Y) = Y_1 - Y_4 + \frac{Y_2 + Y_3}{2} \)

d. \( T_4(Y) = \langle \begin{bmatrix} p & 1 - p & -p & p - 1 \end{bmatrix} , Y \rangle \) for some \( p \in \mathbb{R} \)
Problem 3. Assume that $H \in \{0, 1, 2, 3, 4\}$. For each $H = i$, the transmitter transmits codeword $\mu_i$ and the receiver observes $Y$ where

$$Y = \mu_i + Z \quad \text{and} \quad Z \sim \mathcal{N} \left(0, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right).$$

The codewords are:

$$\mu_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad - \mu_1 = \mu_2 = \begin{bmatrix} 2A \\ 0 \end{bmatrix} \quad - \mu_3 = \mu_4 = \begin{bmatrix} 0 \\ 2A \end{bmatrix}.$$ 

a. Sketch the decision regions assuming that all messages $P_H(i)$ are equally likely.

b. Calculate $P(\text{Error}|H = 0)$ for the decision regions you found in part (a).