
The Moore-Penrose pseudoinverse
CS-526 Learning Theory

Consider a $M \times N$ matrix $A \in \mathbb{C}^{N \times M}$. Its transpose and complex conjugate (also called Hermitian conjugate) is the $N \times M$ matrix $\bar{A}^T$ that we denote A^* . Let $A^\dagger \in \mathbb{C}^{N \times M}$ satisfy the following four conditions:

$$AA^\dagger A = A, \quad A^\dagger AA^\dagger = A^\dagger, \quad (AA^\dagger)^* = AA^\dagger, \quad (A^\dagger A)^* = A^\dagger A.$$

A theorem of Moore and Penrose states that such a matrix always exists and is unique. This matrix is called the Moore-Penrose pseudoinverse. Answer the following questions:

- 1) Let $\Sigma \in \mathbb{C}^{M \times N}$ be a diagonal matrix, that is, $\forall i \neq j : \Sigma_{ij} = 0$ (but you don't necessarily have $M = N$). Show that $\Sigma^\dagger$ is the $N \times M$ diagonal matrix with diagonal entries

$$\forall i \in \{1, 2, \dots, \min\{M, N\}\} : (\Sigma^\dagger)_{ii} = \begin{cases} 1/\Sigma_{ii} & \text{if } \Sigma_{ii} \neq 0; \\ 0 & \text{otherwise.} \end{cases}$$

- 2) Let $A = U\Sigma V^*$ be the singular value decomposition (SVD) of A , that is, both $U \in \mathbb{C}^{M \times M}$ and $V \in \mathbb{C}^{N \times N}$ are unitary matrices and $\Sigma \in \mathbb{R}^{M \times N}$ is a diagonal matrix with real nonnegative diagonal entries (the singular values). Give for $A^\dagger$ an expression that only involves U , V (or their inverse U^* , V^*) and $\Sigma^\dagger$.
- 3) Show that if A has full column rank then $A^\dagger = (A^*A)^{-1}A^*$ and $A^\dagger A = I_{N \times N}$.
- 4) Show that if A has full row rank then $A^\dagger = A^*(AA^*)^{-1}$ and $AA^\dagger = I_{M \times M}$.
- 5) Show that if A is a square matrix with full rank then $A^\dagger = A^{-1}$ is the usual inverse.
- 6) Let A have full column rank and B have full row rank. Check that $(AB)^\dagger = B^\dagger A^\dagger$.