SOLUTION 1. If $H = 0$, we have $Y_2 = Z_1Z_2 = Y_1Z_2$, and if $H = 1$, we have $Y_2 = -Z_1Z_2 = Y_1Z_2$. Therefore, $Y_2 = Y_1Z_2$ in all cases. Now since $Z_2$ is independent of $H$, we clearly have $H \to Y_1 \to (Y_1, Y_1Z_2)$. Hence, $Y_1$ is a sufficient statistic.

SOLUTION 2.

(a) The MAP decoder $\hat{H}(y)$ is given by

$$\hat{H}(y) = \arg \max_i P_{Y|H}(y|i) = \begin{cases} 0 & \text{if } y = 0 \text{ or } y = 1 \\ 1 & \text{if } y = 2 \text{ or } y = 3. \end{cases}$$

$T(Y)$ takes two values with the conditional probabilities

$$P_{T|H}(t|0) = \begin{cases} 0.7 & \text{if } t = 0 \\ 0.3 & \text{if } t = 1 \end{cases} \quad P_{T|H}(t|1) = \begin{cases} 0.3 & \text{if } t = 0 \\ 0.7 & \text{if } t = 1. \end{cases}$$

Therefore, the MAP decoder $\hat{H}(T(y))$ is

$$\hat{H}(T(y)) = \arg \max_i P_{T(Y)|H}(t|i) = \begin{cases} 0 & \text{if } t = 0 \quad (y = 0 \text{ or } y = 1) \\ 1 & \text{if } t = 1 \quad (y = 2 \text{ or } y = 3). \end{cases}$$

Hence, the two decoders are equivalent.

(b) We have

$$\Pr\{Y = 0|T(Y) = 0, H = 0\} = \frac{\Pr\{Y = 0, T(Y) = 0|H = 0\}}{\Pr\{T(Y) = 0|H = 0\}} = \frac{0.4}{0.7} = \frac{4}{7}$$

and

$$\Pr\{Y = 0|T(Y) = 0, H = 1\} = \frac{\Pr\{Y = 0, T(Y) = 0|H = 1\}}{\Pr\{T(Y) = 0|H = 1\}} = \frac{0.1}{0.3} = \frac{1}{3}$$

Thus $\Pr\{Y = 0|T(Y) = 0, H = 0\} \neq \Pr\{Y = 0|T(Y) = 0, H = 1\}$, hence $H \to T(Y) \to Y$ is not true, although the MAP decoders are equivalent.

SOLUTION 3.

(a) The MAP decision rule can always be written as

$$\hat{H}(y) = \arg \max_i f_{Y|H}(y|i)P_H(i)$$

$$= \arg \max_i g_i(T(y))h(y)P_H(i)$$

$$= \arg \max_i g_i(T(y))P_H(i).$$

The last step is valid because $h(y)$ is a non-negative constant which is independent of $i$ and thus does not give any further information for our decision.
(b) Let us define the event $B = \{y : T(y) = t\}$. Then,

$$f_{Y\mid T(Y)}(y\mid i, t) = \frac{f_{Y,T(Y)\mid H}(y, t\mid i)P_H(i)}{f_{T(Y)\mid H}(t\mid i)P_H(i)} = \frac{\Pr\{Y = y, T(Y) = t\mid H = i\}}{\Pr\{T(Y) = t\mid H = i\}} = \frac{\Pr\{Y = y, Y \in B \mid H = i\}}{\Pr\{Y \in B \mid H = i\}}$$

$$= \frac{f_{Y\mid H}(y\mid i)1_B(y)}{\int_B f_{Y\mid H}(y\mid i)dy}.\]

If $f_{Y\mid H}(y\mid i) = g_i(T(y))h(y)$, then

$$f_{Y\mid T(Y)}(y\mid i, t) = \frac{g_i(T(y))h(y)1_B(y)}{\int_B g_i(T(y))h(y)dy} = \frac{g_i(t)h(y)1_B(y)}{g_i(t) \int_B h(y)dy} = \frac{h(y)1_B(y)}{\int_B h(y)dy}.\]

Hence, we see that $f_{Y\mid T(Y)}(y\mid i, t)$ does not depend on $i$, so $H \rightarrow T(Y) \rightarrow Y$.

(c) Note that $P_{Y_k\mid H}(1\mid i) = p_i$, $P_{Y_k\mid H}(0\mid i) = 1 - p_i$ and

$$P_{Y_1, \ldots, Y_n\mid H}(y_1, \ldots, y_n\mid i) = P_{Y_1\mid H}(y_1\mid i) \cdots P_{Y_n\mid H}(y_n\mid i).\]

Thus, we have

$$P_{Y_1, \ldots, Y_n\mid H}(y_1, \ldots, y_n\mid i) = p_i^t(1 - p_i)^{(n-t)},\]

where $t = \sum_k y_k$.

Choosing $g_i(t) = p_i^t(1 - p_i)^{(n-t)}$ and $h(y) = 1$, we see that $P_{Y_1, \ldots, Y_n\mid H}(y_1, \ldots, y_n\mid i)$ fulfills the condition in the question.

(d) Because $Y_1, \ldots, Y_n$ are independent,

$$f_{Y_1, \ldots, Y_n\mid H}(y_1, \ldots, y_n\mid i) = \prod_{k=1}^n \frac{1}{\sqrt{2\pi}} e^{-\frac{(y_k - m_k)^2}{2}} = \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\sum_{k=1}^n \frac{(y_k - m_k)^2}{2}} = \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\sum_{k=1}^n \frac{y_k^2}{2}} e^{nm_i(\frac{1}{2} \sum_{k=1}^n y_k - \frac{m_i}{2})}.\]

Choosing $g_i(t) = e^{nm_i(t - \frac{m_i}{2})}$ and $h(y_1, \ldots, y_n) = \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\sum_{k=1}^n \frac{y_k^2}{2}}$, we see that

$$f_{Y_1, \ldots, Y_n\mid H}(y_1, \ldots, y_n\mid i) = g_i(T(y_1, \ldots, y_n))h(y_1, \ldots, y_n).\]

Hence the condition in the question is fulfilled.
Solution 4.

(a) Since the $X_i$ are i.i.d, the joint probability density (or mass) function is

$$p(x_1, x_2, \ldots, x_n | H = j) = \prod_{i=1}^{n} h(x_i) \exp \left[ c(\theta_j) \sum_{i=1}^{n} T(x_i) - nB(\theta_j) \right].$$

By the Fisher-Neyman factorization theorem, $\sum_{i=1}^{n} T(x_i)$ is a sufficient statistic, where $T(x_i)$ is a sufficient statistic for the random variable $X_i$.

(b) It’s easier to work with single random variables and use the result from (a):

- $p_{X|H}(x|j) = \frac{1}{x!} \exp \left( -\frac{|x - \mu|}{\sigma} \right)$
  
  $h(x) = 1, c(\theta) = -\theta, T(x) = x, B(\theta) = -\frac{1}{\theta}$
  
  By (a), $\sum_{i=1}^{n} x_i$ is a sufficient statistic for $(X_1, X_2, \ldots, X_n)$$

- $p_{X|H}(x|j) = \frac{\lambda^x}{x!} \exp\left( -\frac{\lambda}{x!} \right)$
  
  $h(x) = 1, c(\theta) = \log \theta, T(x) = x, B(\theta) = \log \theta$
  
  By (a), $\sum_{i=1}^{n} x_i$ is a sufficient statistic for $(X_1, X_2, \ldots, X_n)$$

- $p_{X|H}(x|j) = \binom{n}{x} \left( \frac{1}{2} \right)^{n-x} \left( 1 - \frac{1}{2} \right)^x$
  
  $h(x) = \binom{n}{x}, c(\theta) = \log \left( \frac{1}{1-p} \right), B(\theta) = n \log \frac{1}{1-p}$
  
  By (a), $\sum_{i=1}^{n} x_i$ is a sufficient statistic for $(X_1, X_2, \ldots, X_n)$

Solution 5.

(a) Inequality (a) follows from the Bhattacharyya Bound.

Using the definition of DMC, it is straightforward to see that

$$P_{Y|X}(y|c_0) = \prod_{i=1}^{n} P_{Y|X}(y_i|c_{0,i}) \quad \text{and}$$

$$P_{Y|X}(y|c_1) = \prod_{i=1}^{n} P_{Y|X}(y_i|c_{1,i}).$$

(b) follows by substituting the above values in (a).

Equality (c) is obtained by observing that $\sum_{y}$ is the same as $\sum_{y_1, \ldots, y_n}$ (the first one being a vector notation for the sum over all possible $y_1, \ldots, y_n$).

In (c), we see that we want the sum of all possible products. This is the same as summing over each $y_i$ and taking the product of the resulting sum for all $y_i$. This results in equality (d). We obtain (e) by writing (d) in a more concise form.

When $c_{0,i} = c_{1,i}$, $\sqrt{P_{Y|X}(y|c_{0,i}) P_{Y|X}(y|c_{1,i})} = P_{Y|X}(y|c_{0,i})$. Therefore,

$$\sum_{y} \sqrt{P_{Y|X}(y|c_{0,i}) P_{Y|X}(y|c_{1,i})} \leq \sum_{y} P_{Y|X}(y|c_{0,i}) = 1.$$
This does not affect the product, so we are only interested in the terms where \(c_{0,i} \neq c_{1,i}\). We form the product of all such sums where \(c_{0,i} \neq c_{1,i}\). We then look out for terms where \(c_{0,i} = a\) and \(c_{1,i} = b\), \(a \neq b\), and raise the sum to the appropriate power. (Eg. If we have the product \(prpqrpqrr\), we would write it as \(p^3q^2r^4\)). Hence equality (f).

(b) For a binary input channel, we have only two source symbols \(X = \{a, b\}\). Thus,

\[
P_e \leq z^{n(a,b)} z^{n(b,a)} = z^{n(a,b) + n(b,a)} = z^{d_H(c_0, c_1)}.
\]

(c) The value of \(z\) is:

(i) For a binary input Gaussian channel,

\[
z = \int_y \sqrt{f_{Y|X}(y|0)f_{Y|X}(y|1)} \, dy = \exp \left( -\frac{E}{2\sigma^2} \right).
\]

(ii) For the Binary Symmetric Channel (BSC),

\[
z = \sqrt{\Pr\{y = 0|p = 0\} \Pr\{y = 0|p = 1\} + \Pr\{y = 1|p = 0\} \Pr\{y = 1|p = 1\}} = 2\sqrt{\delta(1 - \delta)}.
\]

(iii) For the Binary Erasure Channel (BEC),

\[
z = \sqrt{\Pr\{y = 0|p = 0\} \Pr\{y = 0|p = 1\} + \Pr\{y = E|p = 0\} \Pr\{y = E|p = 1\}}
+ \sqrt{\Pr\{y = 1|p = 0\} \Pr\{y = 1|p = 1\}} = 0 + \delta + 0 = \delta.
\]

**Solution 6.**

\[
P_{00} = \Pr\{(N_1 \geq -a) \cap (N_2 \geq -a)\}
= \Pr\{(N_1 \leq a)\} \Pr\{(N_2 \leq a)\}
= \left[1 - Q\left(\frac{a}{\sigma}\right)\right]^2.
\]

By symmetry:

\[
P_{01} = P_{03} = \Pr\{(N_1 \leq -(2b - a)) \cap (N_2 \geq -a)\}
= \Pr\{N_1 \geq 2b - a\} \Pr\{N_2 \leq a\}
= Q\left(\frac{2b - a}{\sigma}\right) \left[1 - Q\left(\frac{a}{\sigma}\right)\right].
\]

\[
P_{02} = \Pr\{(N_1 \leq -(2b - a)) \cap (N_2 \leq -(2b - a))\}
= \Pr\{N_1 \geq 2b - a\} \Pr\{N_2 \geq 2b - a\}
= \left[Q\left(\frac{2b - a}{\sigma}\right)\right]^2.
\]
\[ P_{0\delta} = 1 - \Pr\{(Y \in \mathcal{R}_0) \cup (Y \in \mathcal{R}_1) \cup (Y \in \mathcal{R}_2) \cup (Y \in \mathcal{R}_3) | c_0 \text{ was sent}\} \\
= 1 - P_{00} - P_{01} - P_{02} - P_{03} \\
= 1 - \left[ 1 - Q\left(\frac{a}{\sigma}\right)\right]^2 - 2Q\left(\frac{2b-a}{\sigma}\right) \left[ 1 - Q\left(\frac{a}{\sigma}\right)\right] - \left[ Q\left(\frac{2b-a}{\sigma}\right)\right]^2 \\
= 1 - \left[ 1 - Q\left(\frac{a}{\sigma}\right) + Q\left(\frac{2b-a}{\sigma}\right)\right]^2. \\
\]
Equivalently,
\[ P_{0\delta} = \Pr\{(N_1 \in [a, 2b-a]) \cup (N_2 \in [a, 2b-a])\} \\
= \Pr\{N_1 \in [a, 2b-a]\} + \Pr\{N_2 \in [a, 2b-a]\} - \Pr\{(N_1 \in [a, 2b-a]) \cap (N_2 \in [a, 2b-a])\} \\
= 2 \left[ Q\left(\frac{a}{\sigma}\right) - Q\left(\frac{2b-a}{\sigma}\right)\right] - \left[ Q\left(\frac{a}{\sigma}\right) - Q\left(\frac{2b-a}{\sigma}\right)\right]^2, \\
\]
which gives the same result as before.