ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

Handout 6 Principles of Digital Communications
Problem Set 3 Mar. 4, 2020

PROBLEM 1. Consider the ternary hypothesis testing problem

$$H_0 : Y = c_0 + Z, \qquad H_1 : Y = c_1 + Z, \qquad H_2 : Y = c_2 + Z,$$

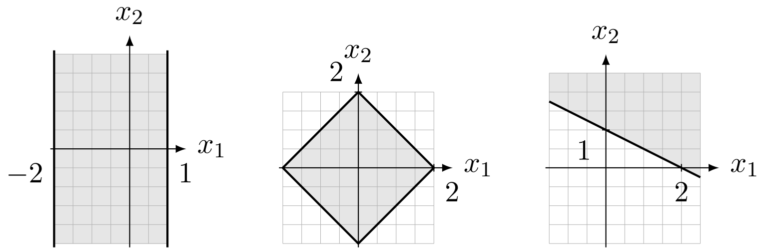
where $Y = [Y_1, Y_2]^\mathsf{T}$ is the two-dimensional observation vector, $c_0 = \sqrt{\mathcal{E}}[1, 0]^\mathsf{T}$, $c_1 = \frac{1}{2}\sqrt{\mathcal{E}}[-1, \sqrt{3}]^\mathsf{T}$, $c_2 = \frac{1}{2}\sqrt{\mathcal{E}}[-1, -\sqrt{3}]^\mathsf{T}$, and $Z = [Z_1, Z_2]^\mathsf{T} \sim \mathcal{N}(0, \sigma^2 I_2)$.

(a) Assuming the three hypotheses are equally likely, draw the optimal decision regions in the (Y_1, Y_2) plane.

(b) Assume now that the apriori probabilities for the hypotheses are $\Pr\{H = 0\} = \frac{1}{2}$, $\Pr\{H = 1\} = \Pr\{H = 2\} = \frac{1}{4}$. Draw the decision regions in the (L_1, L_2) plane where

$$L_i := \frac{f_{Y|H}(Y|i)}{f_{Y|H}(Y|0)}, \quad i = 1, 2.$$

PROBLEM 2. Let $X \sim \mathcal{N}(0, \sigma^2 I_2)$. For each of the three diagrams shown below, express the probability that X lies in the shaded region. You may use the Q function when appropriate.



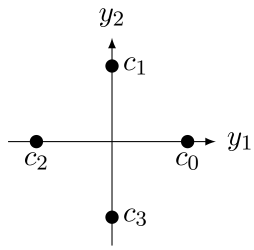
(a) (b) (c)

PROBLEM 3. Let $H \in \{0, 1, 2, 3\}$ and assume that when $H = i$ you transmit the codeword c_i shown in the following diagram. Under $H = i$, the receiver observes $Y = c_i + Z$.

(a) Draw the decoding regions assuming that $Z \sim \mathcal{N}(0, \sigma^2 I_2)$ and that $P_H(i) = 1/4$, $i \in \{0, 1, 2, 3\}$.

(b) Draw the decoding regions (qualitatively) assuming $Z \sim \mathcal{N}(0, \sigma^2 I_2)$ and $P_H(0) = P_H(2) > P_H(1) = P_H(3)$. Justify your answer.

(c) Assume again that $P_H(i) = 1/4$, $i \in \{0, 1, 2, 3\}$ and that $Z \sim \mathcal{N}(0, K)$, where $K = \begin{pmatrix} \sigma^2 & 0 \\ 0 & 4\sigma^2 \end{pmatrix}$. How do you decode now?



PROBLEM 4. The following problem relates to the design of multi-antenna systems. Consider the binary equiprobable hypothesis testing problem:

$$H = 0 : Y_1 = A + Z_1, \quad Y_2 = A + Z_2$$

$$H = 1 : Y_1 = -A + Z_1, \quad Y_2 = -A + Z_2$$

where Z_1, Z_2 are independent Gaussian random variables with different variances $\sigma_1^2 \neq \sigma_2^2$, that is, $Z_1 \sim \mathcal{N}(0, \sigma_1^2)$ and $Z_2 \sim \mathcal{N}(0, \sigma_2^2)$. $A > 0$ is a constant.

(a) Show that the decision rule that minimizes the probability of error (based on the observable Y_1 and Y_2) can be stated as

$$\sigma_2^2 y_1 + \sigma_1^2 y_2 \underset{1}{\overset{0}{\gtrless}} 0$$

(b) Draw the decision regions in the (Y_1, Y_2) plane for the special case where $\sigma_1 = 2\sigma_2$.

(c) Evaluate the probability of the error for the optimal detector as a function of σ_1^2, σ_2^2 and A.

Problem 5. The process of storing and retrieving binary data on a thin-film disk can be modeled as transmitting binary symbols across an additive white Gaussian noise channel where the noise Z has a variance that depends on the transmitted (stored) binary symbol X. The noise has the following input-dependent density:

$$f_Z(z) = \begin{cases} \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{z^2}{2\sigma_1^2}} & \text{if } X = 1 \\ \frac{1}{\sqrt{2\pi\sigma_0^2}} e^{-\frac{z^2}{2\sigma_0^2}} & \text{if } X = 0, \end{cases}$$

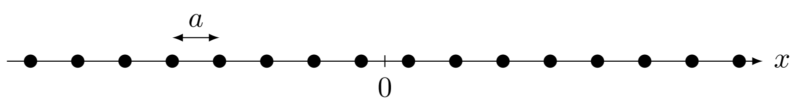
where $\sigma_1 > \sigma_0$. The channel inputs are equally likely.

(a) On the same graph, plot the two possible output probability density functions. Indicate, qualitatively, the decision regions.

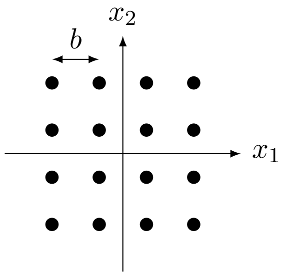
(b) Determine the optimal receiver in terms of σ_0 and σ_1.

(c) Write an expression for the error probability P_e as a function of σ_0 and σ_1.

Problem 6. The following two signal constellations are used to communicate across an additive white Gaussian noise channel. Let the noise variance be σ^2. Each point represents a codeword c_i for some i. Assume each codeword is used with the same probability.



16-PAM



16-QAM

(a) For each signal constellation, compute the average probability of error P_e as a function of the parameters a and b, respectively.

(b) For each signal constellation, compute the average energy per symbol $\mathcal{E}$ as a function of parameters a and b, respectively:

$$\mathcal{E} = \sum_{i=1}^{16} P_H(i) \|c_i\|^2$$

(c) Plot P_e versus $\frac{\mathcal{E}}{\sigma^2}$ for both signal constellations and comment.