ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

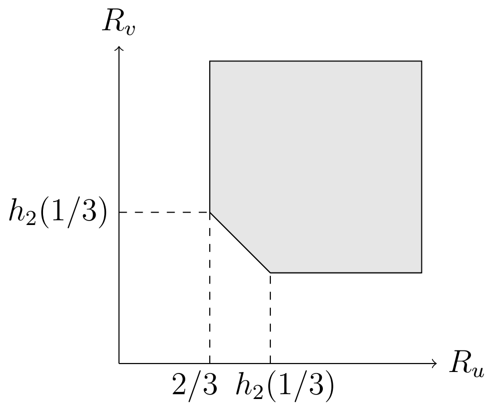
Handout 32 Information Theory and Coding
Solutions to Homework 13 Dec. 24, 2019

Problem 1.

(a) The Slepian-Wolf rate region for (U, V) pair is given as

$$\begin{aligned} R_u &\geq H(U|V) = \log 3 - h_2(1/3) = 2/3 \\ R_v &\geq H(V|U) = \log 3 - h_2(1/3) = 2/3 \\ R_u + R_v &\geq H(UV) = \log 3 \end{aligned}$$

and the region can be drawn as



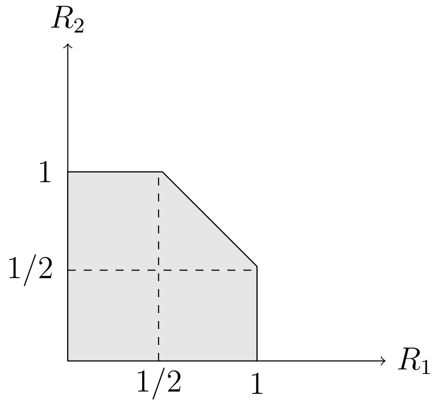
where $h_2(.)$ is the binary entropy function.

(b) The rate region of a MAC with input (X_1, X_2) having a probability distribution $p(x_1x_2) = p(x_1)p(x_2)$ is given by the following polymatroid.

$$R_1 \leq I(X_1; Y|X_2) \tag{1}$$
$$R_2 \leq I(X_2; Y|X_1) \tag{2}$$
$$R_1 + R_2 \leq I(X_1X_2; Y) \tag{3}$$

Note that $I(X_1; Y|X_2) = H(Y|X_2) - H(Y|X_1X_2) = H(Y|X_2) = H(X_1)$. Similarly, $I(X_2; Y|X_1) = H(X_2)$ and $I(X_1X_2; Y) = H(Y) - H(Y|X_1X_2) = H(Y)$. Let $\alpha = \Pr(X_1 = 0)$ and $\beta = \Pr(X_2 = 0)$. Clearly $H(X_1)$ and $H(X_2)$ are maximized when $\alpha = \beta = 1/2$. Moreover for any value of β, $H(Y) = H(X_1 + X_2)$ is a concave function of α and is invariant if we replace α with $1 - \alpha$. Therefore, $\alpha = 1/2$ maximizes $H(Y)$ for any β and by symmetry, $\alpha = \beta = 1/2$ simultaneously maximizes the right hand sides of (1), (2), (3). Then we have the following polymatroid as the capacity region for this MAC.

$$\begin{aligned} R_1 &\leq 1 \\ R_2 &\leq 1 \\ R_1 + R_2 &\leq 3/2. \end{aligned}$$



(c) For this scheme to work, there must exist a (R_u, R_v) pair in the SW region such that $L/N(R_u, R_v)$ belongs to the MAC region. As sum of rates is at least $\log(3)$ in the SW region but at most $3/2$ in the MAC region, L/N can be at most $\frac{3/2}{\log 3} \approx 0.946$. Moreover, it can be seen that for $L/N \leq \frac{3/2}{\log 3}$, the scaled SW region does intersect the MAC region.

(d) With the uncoded scheme, we have $X_1 = U$ and $X_2 = V$ and thus $Y = U + V$. Since U, V are binary and $(U, V) = (0, 1)$ is not possible, the value of Y completely determines (U, V). In this scheme $L/N = 1/1 > 0.946$. Note that in part (c), the maximum value of L/N was found as 0.946. This shows that uncoded schemes can be strictly more efficient in the multi-user settings than coded schemes – something we knew cannot happen in the single user case.

Problem 2.

(a) Note that no matter how user 2 communicates, we can recover X_1 exactly from Y. Let $X_1 \sim \text{Bern}(\alpha)$. Then X_1 can communicate with a rate less than $h_2(\alpha)$. From the side of X_2, the channel is seen as

$$Y = \begin{cases} X_2, & \text{w.p. } \alpha \\ 0, & \text{w.p. } 1 - \alpha \end{cases}$$

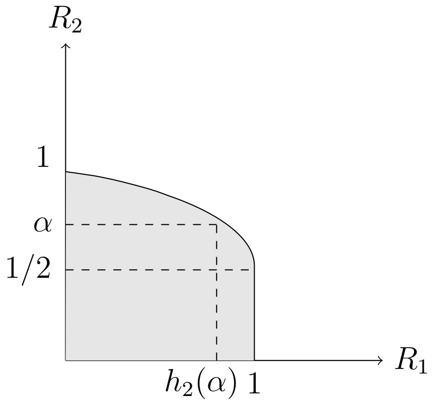
which is essentially a BEC with erasure probability $1 - \alpha$. Therefore, X_2 can communicate with a rate at most α and the following region is obtained.

$$\mathcal{R}(\alpha) = \{(R_1, R_2) : R_1 \leq h_2(\alpha),\ R_2 \leq \alpha\}$$

Note that the constraint for $R_1 + R_2$ is automatically satisfied as $I(X_1 X_2; Y) = H(Y) = \alpha + h_2(\alpha)$. Then the capacity region $\mathcal{R}$ is the convex hull of the union of $\mathcal{R}(\alpha)$'s.

$$\mathcal{R} = \text{conv}\left(\bigcup_{\alpha} \mathcal{R}(\alpha) \right).$$

The region $\mathcal{R}$ is depicted as follows.



(b) The only difference is that the channel from X_2 to Y is a ternary erasure channel. Therefore

$$\mathcal{R}(\alpha) = \{(R_1, R_2) : R_1 \le h_2(\alpha),\ R_2 \le \alpha \log 3\}$$

and the rest is same as part (a).

(c) Taking the logarithm of both sides, we have $\tilde{Y} = \tilde{X}_1 + \tilde{X}_2$, where $\tilde{X}_1 = \log X_1$, $\tilde{X}_2 = \log X_2$, and $\tilde{Y} = \log Y$. Note that $\tilde{X}_1$ and $\tilde{X}_2$ can take values in $\{0, 1\}$ thus this is essentially a binary adder MAC. This capacity region is already found in Problem 1, part (b).