PROBLEM 1.

(a) The Slepian-Wolf rate region for \((U, V)\) pair is given as

\[
\begin{align*}
R_u &\geq H(U|V) = \log 3 - h_2(1/3) = 2/3 \\
R_v &\geq H(V|U) = \log 3 - h_2(1/3) = 2/3 \\
R_u + R_v &\geq H(UV) = \log 3
\end{align*}
\]

and the region can be drawn as

\[
\begin{array}{c}
R_u \\
\hline
2/3 \quad h_2(1/3) \\
\hline
h_2(1/3) \quad R_v
\end{array}
\]

where \(h_2(.)\) is the binary entropy function.

(b) The rate region of a MAC with input \((X_1, X_2)\) having a probability distribution \(p(x_1x_2) = p(x_1)p(x_2)\) is given by the following polymatroid.

\[
\begin{align*}
R_1 &\leq I(X_1; Y|X_2) \\
R_2 &\leq I(X_2; Y|X_1) \\
R_1 + R_2 &\leq I(X_1X_2; Y)
\end{align*}
\]

Note that \(I(X_1; Y|X_2) = H(Y|X_2) - H(Y|X_1X_2) = H(Y|X_2) = H(X_1)\). Similarly, \(I(X_2; Y|X_1) = H(X_2)\) and \(I(X_1X_2; Y) = H(Y) - H(Y|X_1X_2) = H(Y)\). Let \(\alpha = \Pr(X_1 = 0)\) and \(\beta = \Pr(X_2 = 0)\). Clearly \(H(X_1)\) and \(H(X_2)\) are maximized when \(\alpha = \beta = 1/2\). Moreover for any value of \(\beta\), \(H(Y) = H(X_1 + X_2)\) is a concave function of \(\alpha\) and is invariant if we replace \(\alpha\) with \(1 - \alpha\). Therefore, \(\alpha = 1/2\) maximizes \(H(Y)\) for any \(\beta\) and by symmetry, \(\alpha = \beta = 1/2\) simultaneously maximizes the right hand sides of (1), (2), (3). Then we have the following polymatroid as the capacity region for this MAC.

\[
\begin{align*}
R_1 &\leq 1 \\
R_2 &\leq 1 \\
R_1 + R_2 &\leq 3/2
\end{align*}
\]
(c) For this scheme to work, there must exist a \((R_u, R_v)\) pair in the SW region such that 
\(L/N(R_u, R_v)\) belongs to the MAC region. As sum of rates is at least \(\log(3)\) in the
SW region but at most \(3/2\) in the MAC region, \(L/N\) can be at most \(\frac{3/2}{\log 3} \approx 0.946\).
Moreover, it can be seen that for \(L/N \leq \frac{3/2}{\log 3}\), the scaled SW region does intersect
the MAC region.

(d) With the un-coded scheme, we have \(X_1 = U\) and \(X_2 = V\) and thus \(Y = U + V\).
Since \(U, V\) are binary and \((U, V) = (0, 1)\) is not possible, the value of \(Y\) completely
determines \((U, V)\). In this scheme \(L/N = 1/1 > 0.946\). Note that in part (c), the
maximum value of \(L/N\) was found as 0.946. This shows that uncoded schemes can
be strictly more efficient in the multi-user settings than coded schemes – something
we knew cannot happen in the single user case.

**Problem 2.**

(a) Note that no matter how user 2 communicates, we can recover \(X_1\) exactly from \(Y\).
Let \(X_1 \sim \text{Bern}(\alpha)\). Then \(X_1\) can communicate with a rate less than \(h_2(\alpha)\). From the
side of \(X_2\), the channel is seen as
\[
Y = \begin{cases} 
X_2, & \text{w.p. } \alpha \\
0, & \text{w.p. } 1 - \alpha 
\end{cases}
\]
which is essentially a BEC with erasure probability \(1 - \alpha\). Therefore, \(X_2\) can com-
municate with a rate at most \(\alpha\) and the following region is obtained.
\[
\mathcal{R}(\alpha) = \{(R_1, R_2) : R_1 \leq h_2(\alpha), \ R_2 \leq \alpha\}
\]
Note that the constraint for \(R_1 + R_2\) is automatically satisfied as \(I(X_1X_2; Y) = H(Y) = \alpha + h_2(\alpha)\). Then the capacity region \(\mathcal{R}\) is the convex hull of the union of \(\mathcal{R}(\alpha)\)’s.
\[
\mathcal{R} = \text{conv} \left( \bigcup_{\alpha} \mathcal{R}(\alpha) \right).
\]
The region \(\mathcal{R}\) is depicted as follows.
(b) The only difference is that the channel from $X_2$ to $Y$ is a ternary erasure channel. Therefore
\[
R(\alpha) = \{(R_1, R_2) : R_1 \leq h_2(\alpha), \ R_2 \leq \alpha \log 3\}
\]
and the rest is same as part (a).

(c) Taking the logarithm of both sides, we have \(\tilde{Y} = \tilde{X}_1 + \tilde{X}_2\), where \(\tilde{X}_1 = \log X_1\), \(\tilde{X}_2 = \log X_2\), and \(\tilde{Y} = \log Y\). Note that \(\tilde{X}_1\) and \(\tilde{X}_2\) can take values in \(\{0, 1\}\) thus this is essentially a binary adder MAC. This capacity region is already found in Problem 1, part (b).