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Exercise Set 2 : spring 2020  
Calcul Quantique

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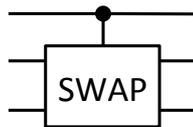
**Exercise 1** *Matrix representations of a few gates*

Consider the following component representation of the canonical computational basis for a quantum bit  $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ,  $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ , and for two quantum bits  $|0\rangle \otimes |0\rangle = |00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ ,  
 $|0\rangle \otimes |1\rangle = |01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ ,  $|1\rangle \otimes |0\rangle = |10\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ ,  $|1\rangle \otimes |1\rangle = |11\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ .

- (a) Give a matrix representation for the following reversible gates : NOT ; CNOT ; CCNOT.  
 (b) Recognize that these are all permutation matrices. What is their inverse matrix ?

**Exercise 2** *Fredkin gate*

The SWAP operation takes two input bits and permutes them :  $\text{SWAP}|b_1, b_2\rangle = |b_2, b_1\rangle$ . The Fredkin gate is a three input controlled SWAP gate and is reversible. The gate swaps the two last bits if the first bit is a 1. Otherwise it leaves the input bits unchanged. One intriguing particularity of the Fredkin gate is that it conserves the number of ones.



- (a) Show that the irreversible gates AND, OR can be represented in a reversible way from the Fredkin gate.  
 (b) Give the matrix representation of the Fredkin gate.  
 (c) Represent the Toffoli (CCNOT) gate in terms of {Fredkin, CNOT}.
- Hint : You can achieve with at most one Fredkin gate and two CNOT gates.

**Exercise 3** *The Mach-Zehnder interferometer.*

Consider the following matrix product  $H(\text{NOT})H$ .

- (a) Is the product unitary? Why?
- (b) Compute the output when the input is  $|0\rangle$ ,  $|1\rangle$ ,  $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$  and  $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ .
- (c) Draw the circuit and interpret it as a quantum interferometer.
- (d) Describe the measurement outcomes at the output of the circuit (interferometer) when we measure in the computational basis.

**Exercise 4** *Production of Bell states*

- a) Compute the four Bell states using the following identity using Dirac's notation. Do not use the component and matrix representations.

$$|B_{xy}\rangle = (\text{CNOT})(H \otimes I)|x\rangle \otimes |y\rangle.$$

where  $x, y \in \{0, 1\}$  and  $|B_{xy}\rangle$  are the Bell states.

- b) Represent the corresponding circuit.
- c) Represent the circuit corresponding to the inverse identity :

$$|x\rangle \otimes |y\rangle = (H \otimes I)(\text{CNOT})|B_{xy}\rangle$$