Exercise 1 Bell states

1) One has to show that $\langle B_{x,y}|B_{x',y'} \rangle = \delta_{x,x'}\delta_{y,y'}$. We show it explicitly for two cases:

$$\langle B_{00}|B_{00} \rangle = \frac{1}{2}(\langle 00| + \langle 11|)(\langle 00| + \langle 11|))$$
$$= \frac{1}{2}(\langle 00|00 \rangle + \langle 00|11 \rangle + \langle 11|00 \rangle + \langle 11|11 \rangle).$$

Now we have

$$\langle 00|00 \rangle = \langle 0|0 \rangle \langle 0|0 \rangle = 1, \langle 00|11 \rangle = \langle 0|1 \rangle \langle 0|1 \rangle = 0,$$
$$\langle 11|00 \rangle = \langle 1|0 \rangle \langle 1|0 \rangle = 0, \langle 11|11 \rangle = \langle 1|1 \rangle \langle 1|1 \rangle = 1.$$

Thus we get that $\langle B_{00}|B_{00} \rangle = \frac{1}{2}(1 + 0 + 0 + 1) = 1$. Now let us consider

$$\langle B_{00}|B_{01} \rangle = \frac{1}{2}(\langle 00| + \langle 11|)(\langle 01| + \langle 10|))$$
$$= \frac{1}{2}(\langle 00|01 \rangle + \langle 00|10 \rangle + \langle 11|01 \rangle + \langle 11|10 \rangle)$$
$$= \frac{1}{2}(0 + 0 + 0 + 0) = 0.$$

2) The proof is by contradiction. Suppose there exists $a_1, b_1$ and $a_2, b_2$ such that

$$|B_{00} \rangle = (a_1 |0 \rangle + b_1 |1 \rangle) \otimes (a_2 |0 \rangle + b_2 |1 \rangle).$$

Then we have

$$\frac{1}{2}(\langle 00 | + \langle 11 |) = a_1 a_2 |00 \rangle + a_1 b_2 |01 \rangle + b_1 a_2 |10 \rangle + a_2 b_2 |11 \rangle.$$ 

Comparing the coefficients of the orthonormal basis, one has

$$\frac{1}{2} = a_1 a_2, \quad \frac{1}{2} = b_1 b_2, \quad a_1 b_2 = 0, \quad b_1 a_2 = 0.$$

The third equality indicates that either $a_1 = 0$ or $b_2 = 0$ (or both). If $a_1 = 0$ we get a contradiction with the first equation. If on the other hand $b_2 = 0$, we get a contradiction with the second one. Therefore, there does not exist $|\psi_1 \rangle$ and $|\psi_2 \rangle$ such that $|B_{00} \rangle$ can be written as $|\psi_1 \rangle \otimes |\psi_2 \rangle$. Therefore, $B_{00}$ is entangled.
3) We have
\[ |\gamma\rangle \otimes |\gamma\rangle = (\cos(\gamma) |0\rangle + \sin(\gamma) |1\rangle) \otimes (\cos(\gamma) |0\rangle + \sin(\gamma) |1\rangle) = \cos^2(\gamma) |00\rangle + \cos(\gamma) \sin(\gamma) |01\rangle + \sin(\gamma) \cos(\gamma) |10\rangle + \sin^2(\gamma) |11\rangle. \]

Similarly, we have
\[ |\gamma\perp\rangle \otimes |\gamma\perp\rangle = = \sin^2(\gamma) |00\rangle - \cos(\gamma) \sin(\gamma) |01\rangle - \sin(\gamma) \cos(\gamma) |10\rangle + \cos^2(\gamma) |11\rangle. \]

Combining the two terms, we find that
\[ |\gamma\rangle \otimes |\gamma\rangle + |\gamma\perp\rangle \otimes |\gamma\perp\rangle = (\cos^2(\gamma) + \sin^2(\gamma)) |00\rangle + (\sin^2(\gamma) + \cos^2(\gamma)) |11\rangle = |00\rangle + |11\rangle \]

and thus
\[ \frac{1}{\sqrt{2}}(|\gamma\rangle \otimes |\gamma\rangle + |\gamma\perp\rangle \otimes |\gamma\perp\rangle) = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = |B_{00}\rangle. \]

4) From the rule of the tensor product
\[ \begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} ac \\ ad \\ bc \\ bd \end{pmatrix}, \]
we obtain the basis states as
\[ |0\rangle \otimes |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |0\rangle \otimes |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \]
\[ |1\rangle \otimes |0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad |1\rangle \otimes |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \]

Thus, we have
\[ |B_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad |B_{01}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \]
\[ |B_{10}\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \quad |B_{11}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}. \]
Exercise 2 Entanglement

1) Our conventions for tensor products are

\[
|0\rangle \otimes |0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad |0\rangle \otimes |1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}
\]

\[
|1\rangle \otimes |0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad |1\rangle \otimes |1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}
\]

Therefore we have

\[
|\Psi\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 0 \\ i \\ 1 \end{pmatrix}
\]

2) Suppose $|\Psi\rangle$ admits a product form $(\alpha |0\rangle + \beta |1\rangle) \otimes (\gamma |0\rangle + \delta |1\rangle)$. We are going to show that it would lead to contradiction. By expanding the tensor product, we have

\[
\alpha \gamma |0,0\rangle + \alpha \delta |0,1\rangle + \beta \gamma |1,0\rangle + \beta \delta |1,1\rangle = \frac{1}{\sqrt{3}} |0,0\rangle + \frac{i}{\sqrt{3}} |1,0\rangle + \frac{1}{\sqrt{3}} |1,1\rangle.
\]

Thus

\[
\alpha \gamma = \frac{1}{\sqrt{3}} \quad (1)
\]

\[
\alpha \delta = 0 \quad (2)
\]

\[
\beta \gamma = \frac{i}{\sqrt{3}}
\]

\[
\beta \delta = \frac{1}{\sqrt{3}} \quad (3)
\]

Eq. (2) suggests that both $\alpha$ and $\gamma$ are non-zero. Then by (3) $\delta$ has to be 0. But this contradicts with (2).

Exercise 3 Entanglement by unitary operations

1) By definition of the tensor product:

\[
(H \otimes I) |x\rangle \otimes |y\rangle = H |x\rangle \otimes I |y\rangle = H |x\rangle \otimes |y\rangle.
\]

Also, one can use that $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ to show that always

\[
H |x\rangle = \frac{1}{\sqrt{2}} (|0\rangle + (-1)^x |1\rangle).
\]
Thus,
\[(H \otimes I) |x\rangle \otimes |y\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |y\rangle + (-1)^x |1\rangle \otimes |y\rangle).\]

Now we apply CNOT. By linearity, we can apply it to each term separately. Thus,
\[(CNOT)(H \otimes I) |x\rangle \otimes |y\rangle = \frac{1}{\sqrt{2}}((CNOT) |0\rangle \otimes |y\rangle + (-1)^x (CNOT) |1\rangle \otimes |y\rangle) \]
\[= \frac{1}{\sqrt{2}}(|0\rangle \otimes |y\rangle + (-1)^x |1\rangle \otimes |y \oplus 1\rangle) \]
\[= |B_{xy}\rangle.\]

2) Let us first start with \(H \otimes I\). We use the rule
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae & af & be & bf \\ ag & ah & bg & bh \\ ce & cf & de & df \\ cg & ch & dg & dh \end{pmatrix},
\]
Thus we have
\[
\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}.
\]

For CNOT, we use the definition:
\[(CNOT) |x\rangle \otimes |y\rangle = |x\rangle \otimes |y \oplus x\rangle,
\]
which implies that the matrix elements are
\[
\langle x'y'|CNOT |xy\rangle = \langle x', y'|x, y \otimes x \rangle = \langle x'|x \rangle \langle y'|y \oplus x \rangle = \delta_{xx'}\delta_{y\oplus x, y'}.
\]

We obtain the following table with columns \(xy\) and rows \(x'y'\):

<table>
<thead>
<tr>
<th></th>
<th>00</th>
<th>01</th>
<th>10</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>00</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>01</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>11</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

For the matrix product \((CNOT)(H \otimes I)\), we find that
\[
(CNOT)H \otimes I = \frac{1}{\sqrt{2}} \begin{pmatrix} I & 0 \\ 0 & X \end{pmatrix} \begin{pmatrix} I & I \\ I & -I \end{pmatrix}
\]
\[= \frac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ X & -X \end{pmatrix},\]
where \( X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \). Thus,

\[
(CNOT)(H \otimes I) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix}.
\]

One can check that for example \(|B_{00}\rangle = (CNOT)(H \otimes I)|0\rangle \otimes |0\rangle\). Finally to check the unitarity, we have to check that \(UU^\dagger = U^\dagger U = I\) for \(U = H \otimes I\), CNOT and \((CNOT)(H \otimes I)\). We leave this to the reader.

3) Let \(U = (CNOT)(H \otimes I)\). We have

\[
|B_{xy}\rangle = U|x\rangle \otimes |y\rangle, \quad \langle x'\rangle = \langle x'\rangle \otimes |y'\rangle U^\dagger.
\]

Thus,

\[
\langle B_{x'y'}|B_{xy}\rangle = \langle x'\rangle \otimes \langle y'\rangle U^\dagger U|x\rangle \otimes |y\rangle
\]

\[
= \langle x'\rangle \otimes \langle y'\rangle |I|x\rangle \otimes |y\rangle
\]

\[
= \langle x'|x\rangle \langle y'|y\rangle = \delta_{xx'} \delta_{yy'}.
\]

**Exercise 4**

*Copying or unitary attack from Eve in BB84*

1) Given Alice sent \(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\), by linearity, the state of the two photons in the lab of Eve just after she made the copying operation is

\[
|\Psi\rangle = U_Z \left( \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes |b\rangle \right)
\]

\[
= U_Z \left( \frac{|0\rangle \otimes |b\rangle}{\sqrt{2}} + \frac{|1\rangle \otimes |b\rangle}{\sqrt{2}} \right)
\]

\[
= \frac{|0\rangle \otimes |0\rangle}{\sqrt{2}} + \frac{|1\rangle \otimes |1\rangle}{\sqrt{2}} = \frac{|00\rangle + |11\rangle}{\sqrt{2}}.
\]

2) In Bob’s lab the outcome is \(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\) with probabilities \(p_\pm = \langle \Psi | \Pi_\pm | \Psi \rangle\), where \(|\Psi\rangle\) is given in Solution 3.1.

Following the hint, we have

\[
\Pi_\pm = (|0\rangle \langle 0| + |1\rangle \langle 1|) \otimes \left( \frac{|0\rangle \pm |1\rangle}{\sqrt{2}} \right) \left( \frac{|0\rangle \pm |1\rangle}{\sqrt{2}} \right)
\]

\[
= (|0\rangle \langle 0| + |1\rangle \langle 1|) \otimes \left( \frac{|0\rangle \langle 0| \pm |0\rangle \langle 1| \pm |1\rangle \langle 0| + |1\rangle \langle 1|}{2} \right)
\]

\[
= \frac{1}{2} \left( |00\rangle \langle 00| \pm |01\rangle \langle 01| \pm |01\rangle \langle 00| + |01\rangle \langle 01| 
\]

\[
+ |10\rangle \langle 10| \pm |11\rangle \langle 11| \pm |11\rangle \langle 10| + |11\rangle \langle 11| \right).
\]
The rest of the calculation is

\[ \Pi_\pm |\Psi\rangle = \frac{1}{2\sqrt{2}} (|00\rangle \pm |01\rangle \pm |10\rangle + |11\rangle), \]

\[ p_\pm = \langle \Psi | \Pi_\pm |\Psi\rangle = \frac{1}{\sqrt{2}} \cdot \frac{1}{2\sqrt{2}} (\langle 00 | + \langle 11 |) (\langle 00 | \pm |01\rangle \pm |10\rangle + |11\rangle) \]

\[ = \frac{1}{4} (\langle 00 |00 \rangle + \langle 11 |11 \rangle) \]

\[ = \frac{1}{2}. \]