In this exercise we illustrate a toy model description of the “orbital” state of a photon and its manipulation by mirrors. We suppose that a photon can travel only along the “vertical” and “horizontal” directions and represent its general state by a quantum bit

$$\alpha |H\rangle + \beta |V\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

where $|H\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|V\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Recall that $\alpha, \beta \in \mathbb{C}$ and $\alpha^* \alpha + \beta^* \beta = 1$.

![Figure 1: Possible preparation directions and states](image)

1) Write down the “Bra” in Dirac and usual vector notations associated to the “Ket” $\alpha |H\rangle + \beta |V\rangle$.

2) Compute the scalar product (or Bracket) for the two kets $\alpha |H\rangle + \beta |V\rangle$ and $\gamma |H\rangle + \delta |V\rangle$ in Dirac and vector notations. In particular, check $\langle H | V \rangle = \langle V | H \rangle = 0$ and $\langle H | H \rangle = \langle V | V \rangle = 1$.

3) A mirror like Figure 2 makes the transitions $|H\rangle \rightarrow i |V\rangle$ and $|V\rangle \rightarrow i |H\rangle$. In quantum physics the mirror operation is described by a matrix $R = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$. Verify $R^\dagger R = RR^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ where $R = R^{\top,*}$ (transpose and complex conjugate).

A photon in the state $\alpha |H\rangle + \beta |V\rangle$ is incident on the mirror and is reflected. Compute the output state $R (\alpha |H\rangle + \beta |V\rangle)$ in Dirac and vector notations. Make a picture of the photon and mirror.
4) A semi-transparent mirror is described by a matrix
\[ S = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & i \\
i & 1
\end{pmatrix}. \]
Verify \( S^\dagger S = S S^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \) Compute \( S \ket{H}, S \ket{V} \) and \( S (\alpha \ket{H} + \beta \ket{V}) \). Make pictures analogous to Figure 2 to get an intuition of these operations.

5) Consider the following experiment where \( D \) is a photo-detection.

\[
\text{Incoming photon in state } \alpha \ket{H} + \beta \ket{V} \rightarrow \text{Semi-transparent mirror} \rightarrow \text{Outgoing photon in state } S (\alpha \ket{H} + \beta \ket{V}) \rightarrow D
\]

Here we will assume \( \alpha \) and \( \beta \) are real. Compute the probability of detecting a photon in \( D \).

6) Now we consider the following set-up which constitutes the Mach–Zehnder interferometer.

\[
\text{The incoming photon has the state } \ket{H}. \text{ Compute the final state just after the second semi-transparent mirror. Then compute the probability of detecting the photon in } D_1 \text{ and } D_2. \text{ What would you expect to find if the photons were “classical balls”?}
\]