ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

Handout 35 | Principles of Digital Communications
Final exam | June 24, 2019

3 problems, 30 points
180 minutes
2 double-sided A4 sheets of notes allowed.

Good Luck!

PLEASE WRITE YOUR NAME ON EACH SHEET OF YOUR ANSWERS.

PLEASE WRITE THE SOLUTION OF EACH PROBLEM ON A SEPARATE SHEET.

PROBLEM 1. (8 points) In a binary hypothesis testing problem, the hypothesis H can take two possible values $\{0,1\}$, with apriori probabilities $P_H(0), P_H(1)$. We want to guess the value of H given an observation $Y \in \mathbb{R}$. The conditional distribution of Y given $H = i$ is denoted by $f_{Y|H}(y|i)$.
Assume we are given the possibility to *not* make a decision, i.e. have a decision rule with output $\hat{H}(Y) \in \{0, 1, ?\}$, where ? denotes no decision. In this situation we distinguish two kinds of undesirable events:

(1) $\hat{H} = ?$;

(2) $\hat{H} \neq ?$ and $\hat{H} \neq H$.

Let $P_?$ and P_e denote the probabilities of these events. A reasonable benchmark to measure the quality of a decision rule is the weighted sum $P_e + \alpha P_?$, for some $\alpha \geq 0$.

(a) (2pts) Let $\mathcal{R}_0, \mathcal{R}_1, \mathcal{R}_?$ denote the observations for which $\hat{H}(y)$ gives $0, 1$ and ?, respectively. Show that

$$P_e + \alpha P_? = \int_{\mathcal{R}_1} b_0(y)\, dy + \int_{\mathcal{R}_0} b_1(y)\, dy + \int_{\mathcal{R}_?} b_?(y)\, dy,$$

where $b_0(y), b_1(y), b_?(y)$ are defined as

$$\begin{aligned} b_0(y) &= P_H(0) f_{Y|H}(y|0), \\ b_1(y) &= P_H(1) f_{Y|H}(y|1), \\ b_?(y) &= \alpha \left(P_H(0) f_{Y|H}(y|0) + P_H(1) f_{Y|H}(y|1) \right). \end{aligned}$$

(b) (2pts) Consider the following decision rule:

$$\hat{H}(y) = \begin{cases} 1 & b_0(y) = b(y), \\ 0 & b_1(y) = b(y), \\ ? & b_?(y) = b(y), \end{cases}$$

where $b(y) = \min(b_0(y), b_1(y), b_?(y))$. Show that this decision rule minimizes $P_e + \alpha P_?$.
Hint: By (a), $P_e + \alpha P_? \geq \int_y b(y)\, dy$ for *any* decision rule.

(c) (2pts) Assume $\alpha \in]0, \frac{1}{2}[$ and let $L(y) = \frac{P_H(0) f_{Y|H}(y|0)}{P_H(1) f_{Y|H}(y|1)}$. Show that the decision rule in (b) is equivalent to the following likelihood ratio test:

$$\hat{H}(y) = \begin{cases} 0 & L(y) > \frac{1-\alpha}{\alpha}, \\ 1 & L(y) < \frac{\alpha}{1-\alpha}, \\ ? & \text{otherwise.} \end{cases}$$

(d) (2pts) Assume $\alpha \in]0, \frac{1}{2}[$. Suppose $Z \sim \mathcal{N}(0, \sigma^2)$ independent of H, $P_H(0) = P_H(1)$, and observable Y is given by

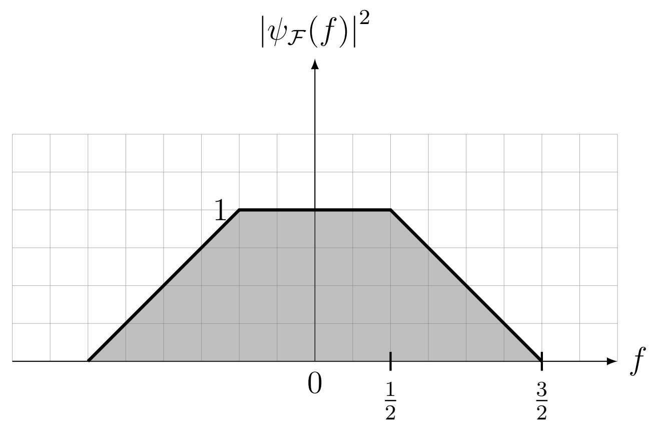
$$Y = \begin{cases} 1 + Z & H = 0, \\ -1 + Z & H = 1. \end{cases}$$

Sketch the decision regions $\mathcal{R}_0, \mathcal{R}_1, \mathcal{R}_?$.

Problem 2. (12 points) A transmitter uses the following waveforms $\mathcal{S} = \{w_0, w_1, w_2, w_3\}$ to communicate one of four equally likely messages over an AWGN channel of power spectral density $N_0/2$:

$$w_0(t) = 0, \qquad w_1(t) = \psi(t), \qquad w_2(t) = \psi(t-T), \qquad w_3(t) = w_1(t) + w_2(t),$$

where $\psi(t)$ is a signal with the following squared Fourier transform:



(a) (2pts) How should T be chosen for $w_1(t), w_2(t)$ to be orthogonal?

(b) (2pts) Design a receiver that uses the smallest number of filters to optimally detect $\mathcal{S}$.

(c) (2pts) Apply a translation to $\mathcal{S}$ to obtain a minimum energy signal set $\tilde{\mathcal{S}}$. Sketch the new signal set in the coordinate system spanned by $\{\psi(t), \psi(t-T)\}$.

(d) (2pts) Explain how to modify the receiver designed in (b) to obtain a MAP receiver for $\tilde{\mathcal{S}}$.

(e) (2pts) Find the error probability of the system when decoding $\mathcal{S}$ with the receiver in (b).

(f) (2pts) Find the error probability of the system when decoding $\tilde{\mathcal{S}}$ with the receiver in (b).

PROBLEM 3. (10 points) Consider the following convolutional code that transforms a sequence of data bits $\{b_i \in \pm 1,\ i \geq 1\}$ to coded symbols $\{x_i \in \pm 1,\ i \geq 1\}$:

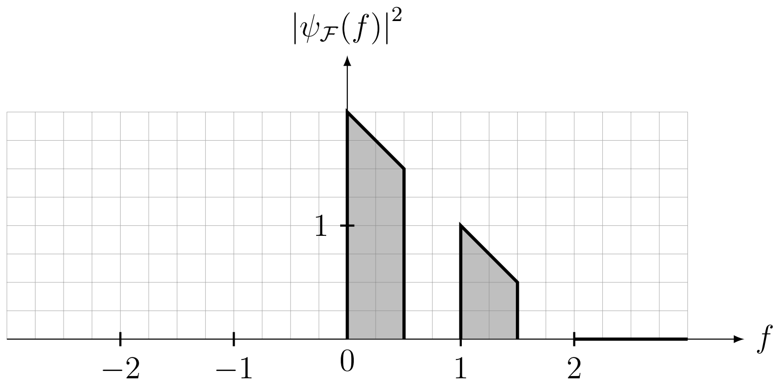
$$\begin{aligned} x_{2i-1} &= b_i, \\ x_{2i} &= b_i b_{i-2}. \end{aligned} \tag{1}$$

The coded symbols are transmitted over an AWGN channel as

$$w(t) = \sqrt{\mathcal{E}} \sum_k x_k \psi(t-k),$$

where $\psi(t)$ is a real-valued pulse such that $\{\psi(t-k),\ k \in \mathbb{Z}\}$ forms an orthogonal collection.

(a) (2pts) Complete the missing sections of the plot below:



(b) (2pts) Find the autocorrelation function $K_X[k] = \mathbb{E}[X_i X_{i+k}]$ of the transmitted symbols.

(c) (2pts) Draw the state diagram description of the convolutional code.

(d) (2pts) Let $c_1, c_2, \ldots$ and $d_1, d_2, \ldots$ denote the *even*-indexed and *odd*-indexd inputs bits, i.e. $c_k = b_{2k},\ d_k = b_{2k-1},\ k \geq 1$. Show how to implement the encoder using two parallel two-state encoders.

(e) (2pts) We encode four bits b_1, b_2, b_3, b_4 using (1) assuming the initial and final states of the encoder are all-ones. At the receiver the noisy signal $R(t)$ is sent through the matched filter $\psi(-t)$, the output of which is sampled at integer instants to form the discrete time sequence $\{Y_k \in \mathbb{R},\ k \geq 1\}$. Given the realizations below (which are integer-valued to ease computations):

Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	Y_7	Y_8	Y_9	Y_{10}	Y_{11}	Y_{12}
12	11	1	−17	5	−2	6	−17	−3	3	6	−10,

find the maximally likely sequence $\hat{b}_1, \hat{b}_2, \hat{b}_3, \hat{b}_4$.

Hint: Using the implementation in (d) may simplify computations.