ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

Handout 33 | Principles of Digital Communications
Mock Final Exam (Final of 2017) | May 29, 2019

3 problems, 55 points, 180 minutes, closed book
one handwritten (double-sided) A4 page of summary allowed
Good luck!

Please write your name on each sheet of your answers

Please write the solution of each problem on a separate sheet

Problem 1. (15 points)

Suppose the random process $X(t)$ is generated as

$$X(t) = \sum_i X_i \psi(t - i - \Theta),$$

where $\{X_i : i \in \mathbb{Z}\}$ is the input to the waveform former, Θ is uniformly distributed in $[0, 1]$ and $\psi(t)$ is given by

$$\psi(t) = \operatorname{sinc}(t) \cdot \operatorname{tri}(t),$$

where

$$\operatorname{sinc}(t) = \frac{\sin(\pi t)}{\pi t},$$
$$\operatorname{tri}(t) = (1 - |t|)\mathbb{1}\{|t| < 1\}.$$

The process $X(t)$ passes through a filter with impulse response

$$h(t) = \delta(t) - 2\delta(t - 1/4) + \delta(t - 1/2),$$

the output of which is denoted by $Y(t)$.

(a) (4 pts) Suppose $X_i = \sqrt{\mathcal{E}} D_i$ where $\{D_i : i \in \mathbb{Z}\}$ are i.i.d, $\{+1, -1\}$-valued, zero-mean random variables. Find $S_Y(f)$, the power spectrum of $Y(t)$. At what frequencies is $S_Y(f)$ equal to zero?

Hint: $\operatorname{tri}(t)$ is the convolution of $\operatorname{rect}(t) = \mathbb{1}\{|t| < 1/2\}$ with itself, and $\operatorname{rect}(t)$ and $\operatorname{sinc}(f)$ are Fourier transform pairs. You can express your answer in terms of

$$a(x) = \int_{x-1/2}^{x+1/2} \operatorname{sinc}^2(u)du.$$

(b) (4 pts) With D_i as above, suppose now $X_i = s[D_i + \alpha D_{i-4}]$, where s is a scaling factor to ensure that $\mathbb{E}[X_i^2]$ is the same as in (a). Find the value of s, and redo (a) with this choice of X_i.

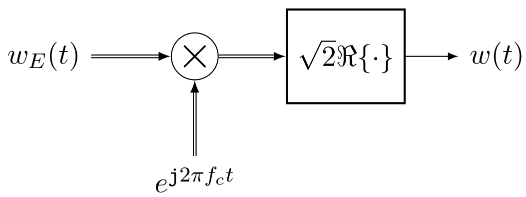
(c) (4 pts) Again with D_i as above, suppose now $X_i = sD_iD_{i-1}(D_i + D_{i-1})$, and redo (b).

(d) (3 pts) Suppose we are required to make $S_Y(1) = 0$. Does the method in (c) satisfy this requirement? Is there a choice of α in (b) to satisfy this requirement? Explain your answers, and find this choice of α if it exists.

Problem 2. (18 points) In a 4-PSK passband communication system, given the i.i.d. data sequence $X_j \in \{+1, -1, +\mathrm{j}, -\mathrm{j}\}$, the transmitter creates the complex-valued baseband waveform $w_E(t)$ as

$$w_E(t) = \sum_j X_j \operatorname{sinc}(t-j).$$

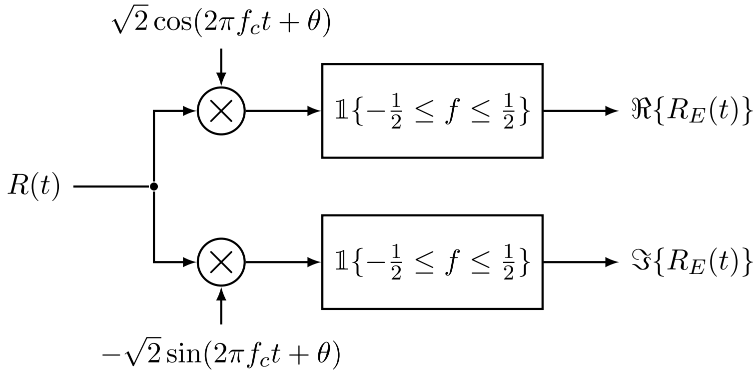
The baseband signal $w_E(t)$ is, subsequently, up-converted to a passband signal $w(t)$ around the carrier frequency $f_c \gg \frac{1}{2}$ as follows:



The receiver, *which is only capable of performing real-valued operations*, observes the signal $R(t)$ that is given by

$$R(t) = A \cdot w(t) + N(t),$$

where A is the amplification gain, and $N(t)$ is a white Gaussian noise of power spectral density $\frac{N_0}{2}$. After observing $R(t)$, the receiver down-converts it to baseband. However, there is a phase difference of θ between the oscillators of the transmitter and the receiver:



Let us first assume that A is known to the receiver.

(a) (1 pt) If $w(t)$ satisfies $w_{\mathcal{F}}(f_c - f) = w_{\mathcal{F}}(f_c + f)^*$ for $f \in (-\frac{1}{2}, \frac{1}{2})$, what can we say about the values of X_j?

(b) (1 pt) If $w(t)$ satisfies $w_{\mathcal{F}}(f_c - f) = -w_{\mathcal{F}}(f_c + f)^*$ for $f \in (-\frac{1}{2}, \frac{1}{2})$, what can we say about the values of X_j?

(c) (4 pts) Assuming that $\theta = 0$ and that A is known to the receiver, complete the block-diagram of the receiver to form a two-dimensional sufficient statistic for estimating the data symbols X_j. Do you need any additional filters?

(d) (4 pts) Determine the ML decision rule (for estimating the data symbols X_j), sketch the decision regions, and express the probability of error of the receiver in terms of the gain A and Q functions (assuming that $\theta = 0$, and that A is known to the receiver).

(e) (4 pts) Assume that A is a random variable that is distributed as follows:

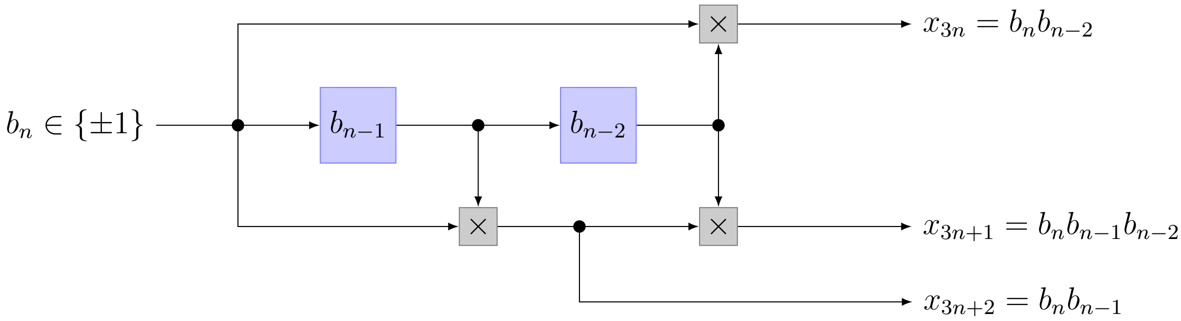
$$A = \begin{cases} \sqrt{\frac{1}{2}} & \text{with probability } \frac{1}{2}, \\ \sqrt{\frac{3}{2}} & \text{with probability } \frac{1}{2}. \end{cases}$$

If A were unknown to the receiver, would you change anything in the receiver of part (d)? Express the average probability of error of the receiver in terms of Q functions (still assuming that $\theta = 0$).

Hint: Do the ML decision regions in (d) depend on the value of A?

(f) (4 pts) Now assume that $\theta \in (\frac{\pi}{4}, \frac{3\pi}{4})$ and is unknown to the receiver. Assume also that A is as in part (e). Express the average probability of error of the receiver of part (d) as a function of θ.

Problem 3. (22 points) Consider the following convolutional encoder:



We assume the encoder starting state is $(1, 1)$.

(a) (4 pts) Draw the state diagram and the detour flow graph.

(b) (2 pts) Given the input sequence $(1, -1, -1, 1, 1)$ find the corresponding encoder output sequence $(x_1, \cdots, x_{15})$.

We want to use the above convolutional encoder in order to send information reliably across the channel $\mathcal{C}$ whose input set is $\{\pm 1\}$, output set is $\{\pm 1, ?\}$ and described by

$$\begin{aligned} \mathbb{P}_{Y|X}(1|1) = \mathbb{P}_{Y|X}(-1|-1) &= 1 - \epsilon \\ \mathbb{P}_{Y|X}(?|1) = \mathbb{P}_{Y|X}(?|-1) &= \frac{\epsilon}{2} \\ \mathbb{P}_{Y|X}(-1|1) = \mathbb{P}_{Y|X}(1|-1) &= \frac{\epsilon}{2}. \end{aligned}$$

So given an input x_i the channel will erase it with probability $\frac{\epsilon}{2}$, flip it with probability $\frac{\epsilon}{2}$ and keep it unchanged with probability $1 - \epsilon$, where we assume $\epsilon < \frac{2}{3}$.

(c) (3 pts) Assuming equiprobable priors, find the Bhattacharyya bound z for this channel.

(d) (4 pts) Derive an upper bound on the bit error probability in terms of z.

(e) (4 pts) Derive the branch metric for a maximum-likelihood Viterbi decoder and specify whether the decoder chooses the path with largest or smallest path metric.

(f) (5 pts) The channel outputs the sequence

$$(y_1, \cdots, y_{15}) = (1, -1, 1, ?, -1, -1, ?, -1, 1, -1, 1, -1, -1, -1, 1).$$

Given that the decoder knows that the last two information bits are $+1$, find the maximum-likelihood estimate of the information sequence. Compare your answer with the encoder input sequence in question (b).

Hint: To simplify the computation you may wish to denote $\log(1-\epsilon)$ by $-u$ and $\log\left(\frac{\epsilon}{2}\right)$ by $-v$, and express the branch metrics in term of u and v. Note that $0 < u < v$ when $\epsilon < 2/3$.