

# ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

## Handout 21

Solutions to Problem Set 8

Principles of Digital Communications

Apr. 17, 2019

SOLUTION 1. First we compute  $T_s$ , which is the duration of one bit:

$$T_s = \frac{1}{1 \text{ Mbps}} = 10^{-6} \text{ s.}$$

Now, we can calculate the energy of the signal (i.e. the energy per bit), which is the same for every  $j$ :

$$\mathcal{E}_b = b^2 T_s.$$

The bit error probability is given by  $Q\left(\frac{\sqrt{\mathcal{E}_b}}{\sigma}\right)$ . In our case  $\sigma = \sqrt{N_0/2} = 10^{-1}$ , thus we need to solve

$$10^{-5} = Q\left(\frac{10^{-3} \times b}{10^{-1}}\right) = Q(10^{-2} \times b),$$

hence  $b = Q^{-1}(10^{-5}) \times 10^2 \approx 426.5$ .

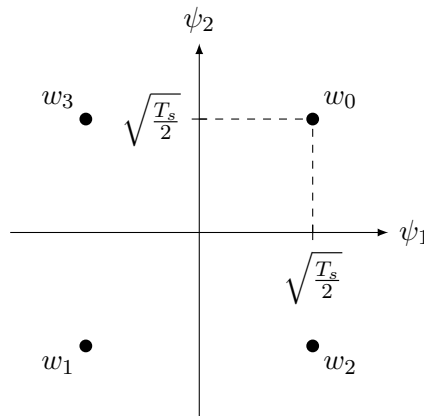
SOLUTION 2.

(a) There are various possibilities to choose an orthogonal basis. One is  $\phi_1(t) = \frac{w_0(t)}{\|w_0\|} = \sqrt{\frac{1}{T_s}} w_0(t)$  and  $\phi_2(t) = \frac{w_2(t)}{\|w_2\|} = \sqrt{\frac{1}{T_s}} w_2(t)$ . Another choice, that we prefer and will be our choice in this solution is

$$\begin{aligned}\psi_1(t) &= \sqrt{\frac{2}{T_s}} \mathbb{1}_{[0, \frac{T_s}{2}]}(t) \\ \psi_2(t) &= \sqrt{\frac{2}{T_s}} \mathbb{1}_{[\frac{T_s}{2}, T_s]}(t).\end{aligned}$$

With the latter choice the signal space is

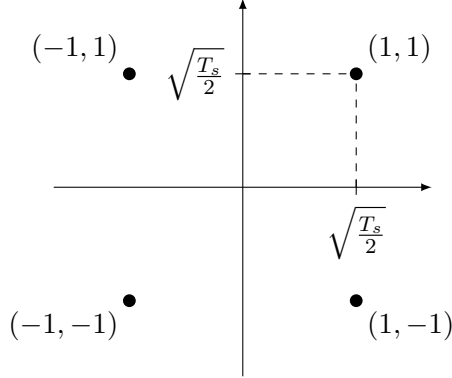
$$\begin{aligned}w_0 &= \sqrt{\frac{T_s}{2}} (1, 1)^\top & w_2 &= \sqrt{\frac{T_s}{2}} (1, -1)^\top \\ w_1 &= \sqrt{\frac{T_s}{2}} (-1, -1)^\top & w_3 &= \sqrt{\frac{T_s}{2}} (-1, 1)^\top\end{aligned}$$



(b)  $U_0 \in \{\pm 1\}$  and  $U_1 \in \{\pm 1\}$  are mapped into

$$U_0 \sqrt{\frac{T_s}{2}} \psi_1(t) + U_1 \sqrt{\frac{T_s}{2}} \psi_2(t).$$

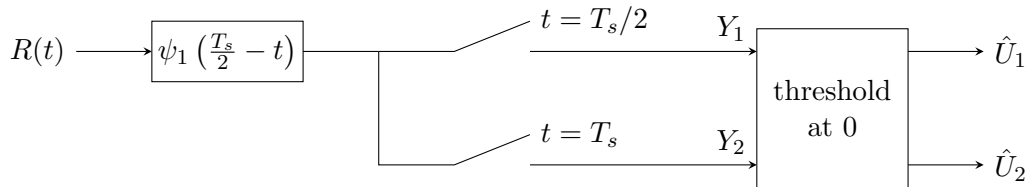
The mapping is shown below:



The mapping is such that neighboring points differ by one bit. This minimizes the bit-error probability since when we make an error chances are that we choose a neighbor of the correct symbol. Notice that we may decode each bit independently. In fact the first bit is decoded to a 1 iff the observation is to the right of the vertical axis and the second bit is 1 iff it is above the horizontal axis. The bit error probability is therefore

$$P_b = Q\left(\frac{\sqrt{T_s/2}}{\sqrt{N_0/2}}\right) = Q\left(\sqrt{\frac{T_s}{N_0}}\right).$$

(c) Notice that  $\psi_2(t) = \psi_1(t - \frac{T_s}{2})$ . Hence one matched filter is enough. The receiver block diagram is:



(d)  $\mathcal{E}_b = \frac{\mathcal{E}_s}{2} = \frac{T_s}{2}$  and the power is  $\frac{\mathcal{E}_s}{T_s} = 1$ .

SOLUTION 3.

(a) Using the identity  $\cos^2(a) = \frac{1}{2}[1 + \cos(2a)]$ , the average energy can be computed as

$$\begin{aligned} \int_{-\infty}^{\infty} |w_i(t)|^2 dt &= \frac{2\mathcal{E}}{T} \int_0^T \cos^2(2\pi(f_c + i\Delta f)t) dt \\ &= \frac{2\mathcal{E}}{T} \left[ \frac{t}{2} + \frac{\sin(4\pi(f_c + i\Delta f)t)}{8\pi(f_c + i\Delta f)} \right]_0^T \\ &= \mathcal{E} \left[ 1 + \frac{\sin(4\pi i\Delta f T)}{4\pi(f_c + i\Delta f)} \right] \approx \mathcal{E}. \end{aligned} \quad (*)$$

The last approximation follows since  $f_c \gg \Delta f$  implies the second term in the square brackets is negligible.

(b) Orthogonality requires

$$\mathcal{E} \frac{2}{T} \int_0^T \cos(2\pi(f_c + i\Delta f)t) \cos(2\pi(f_c + j\Delta f)t) dt = 0,$$

for every  $i \neq j$ . Using the trigonometric identity  $\cos(\alpha) \cos(\beta) = \frac{1}{2} \cos(\alpha + \beta) + \frac{1}{2} \cos(\alpha - \beta)$ , an equivalent condition is

$$\frac{\mathcal{E}}{T} \int_0^T [\cos(2\pi(i - j)\Delta f t) + \cos(2\pi(2f_c + (i + j)\Delta f)t)] dt = 0.$$

Integrating we obtain

$$\frac{\mathcal{E}}{T} \left[ \frac{\sin(2\pi(i - j)\Delta f T)}{2\pi(i - j)\Delta f} + \frac{\sin(2\pi(2f_c + (i + j)\Delta f)T)}{2\pi(2f_c + (i + j)\Delta f)} \right] = 0.$$

As  $f_c T$  is assumed to be an integer, the result can be simplified to

$$\frac{\mathcal{E}}{T} \left[ \frac{\sin(2\pi(i - j)\Delta f T)}{2\pi(i - j)\Delta f} + \frac{\sin(2\pi(i + j)\Delta f T)}{2\pi(2f_c + (i + j)\Delta f)} \right] = 0.$$

As  $i$  and  $j$  are integer, this is satisfied for  $i \neq j$  if and only if  $2\pi\Delta f T$  is an integer multiple of  $\pi$ . Hence, we obtain the minimum value of  $\Delta f$  if  $2\pi\Delta f T = \pi$  which gives  $\Delta f = \frac{1}{2T}$ . Note that once  $\Delta f$  is an integer multiple of  $\frac{1}{2T}$  the approximate equality in (\*) will be exact.

(c) Proceeding similarly, we will have orthogonality if and only if

$$\begin{aligned} \frac{\mathcal{E}}{T} \left[ \frac{\sin(2\pi(i - j)\Delta f T + \theta_i - \theta_j) - \sin(\theta_i - \theta_j)}{2\pi(i - j)\Delta f} \right. \\ \left. + \frac{\sin(2\pi(i + j)\Delta f T + \theta_i + \theta_j) - \sin(\theta_i + \theta_j)}{2\pi(2f_c + (i + j)\Delta f)} \right] = 0. \end{aligned}$$

In this case we see that both parts become zero if and only if  $2\pi\Delta f T$  is an even multiple of  $\pi$ , meaning that the smallest  $\Delta f$  is  $\Delta f = \frac{1}{T}$  which is twice the minimum frequency separation needed in the previous part. Hence, the cost of phase uncertainty is a bandwidth expansion by a factor of 2.

(d) The condition for essential orthogonality is that

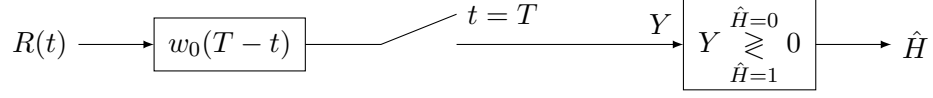
$$\begin{aligned} \frac{\mathcal{E}}{T} \left[ \frac{\sin(2\pi(i - j)\Delta f T + \theta_i - \theta_j) - \sin(\theta_i - \theta_j)}{2\pi(i - j)\Delta f} \right] \\ + \frac{\mathcal{E}}{T} \left[ \frac{\sin(2\pi(2f_c + (i + j)\Delta f)T + \theta_i + \theta_j) - \sin(\theta_i + \theta_j)}{2\pi(2f_c + (i + j)\Delta f)} \right] \end{aligned}$$

is small compared to the signal's energy  $\mathcal{E}$ . The first term vanishes if  $\Delta f = \frac{1}{T}$ . The second term is very small compared to  $\mathcal{E}$  if  $f_c T \gg 1$ .

(e) We have  $m$  signals separated by  $\Delta f$ . The approximate bandwidth is  $m\Delta f$ . This means bandwidth  $\frac{2^k}{2T}$  without random phase, and bandwidth  $\frac{2^k}{T}$  with random phase. We see that in both cases,  $WT$  is proportional to  $2^k$ , i.e. it grows exponentially with  $k$ .

SOLUTION 4.

(a) The block diagram is shown below:



(b) Given  $A = a$ , the distance of signals is  $2a\sqrt{\mathcal{E}_b}$ , hence

$$P_e(a) = Q\left(a\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right).$$

(c)

$$P_f = \mathbb{E}[P_e(a)] = \int_0^\infty Q\left(a\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right) 2ae^{-a^2} da.$$

We integrate by parts, noting that  $\int 2ae^{-a^2} da = -e^{-a^2}$ :

$$P_f = -Q\left(a\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right) e^{-a^2} \Big|_0^\infty + \int_0^\infty Q'\left(a\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right) e^{-a^2} da.$$

Taking the derivative of an integral with respect to the lower boundary gives the negative of the value of the integrand evaluated at the lower boundary, i.e.,

$$Q'(x) = -\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

Thus, for the derivative of  $Q\left(a\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)$  with respect to  $a$ , we can write

$$\frac{d}{da} Q\left(a\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right) = -\frac{1}{\sqrt{2\pi}} e^{-\frac{a^2 \mathcal{E}_b}{N_0}} \sqrt{\frac{2\mathcal{E}_b}{N_0}}.$$

Plugging this in, we find

$$P_f = \frac{1}{2} - \int_0^\infty \frac{1}{\sqrt{2\pi}} \sqrt{\frac{2\mathcal{E}_b}{N_0}} e^{-a^2 \left(\frac{\mathcal{E}_b}{N_0} + 1\right)} da,$$

which we now reshape to make it an integral over a Gaussian density, as follows:

$$P_f = \frac{1}{2} - \sqrt{\frac{2\mathcal{E}_b}{N_0}} \frac{1}{\sqrt{2\left(\frac{\mathcal{E}_b}{N_0} + 1\right)}} \int_0^\infty \frac{1}{\sqrt{\frac{\pi}{\left(\frac{\mathcal{E}_b}{N_0} + 1\right)}}} \exp\left(-\frac{a^2}{2\left(\frac{1}{\frac{\mathcal{E}_b}{N_0} + 1}\right)}\right) da.$$

Now, it is clear that the integral evaluates to one half (since the integral is only over half of the real line), and we find

$$P_f = \frac{1}{2} - \frac{1}{2} \sqrt{\frac{\mathcal{E}_b/N_0}{1 + \mathcal{E}_b/N_0}} = \frac{1}{2} \left(1 - \sqrt{\frac{\mathcal{E}_b/N_0}{1 + \mathcal{E}_b/N_0}}\right).$$

(d) Let  $\sigma = \frac{1}{\sqrt{2}}$ , then

$$m = \mathbb{E}[A] = \int_0^\infty 2a^2 e^{-a^2} da = 2\sqrt{\pi} \int_0^\infty a^2 \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{a^2}{2\sigma^2}} da = \sqrt{\pi}\sigma^2 = \frac{\sqrt{\pi}}{2}.$$

Thus, using the formula from part (b):

$$P_e(m) = Q\left(m\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right) = Q\left(\sqrt{\frac{\pi}{2}}\sqrt{\frac{\mathcal{E}_b}{N_0}}\right).$$

For the given example we get

$$\frac{\mathcal{E}_b}{N_0} = \frac{2(Q^{-1}(10^{-5}))^2}{\pi} \approx 10.6 \text{ dB}.$$

For the fading we use the result of part (c) to get

$$\frac{\mathcal{E}_b}{N_0} = \frac{(1 - 2 \cdot 10^{-5})^2}{1 - (1 - 2 \times 10^{-5})^2} \approx 44 \text{ dB}.$$

The difference is quite significant! It is clear that this behaviour is fundamentally different from the non-fading case.

SOLUTION 5.

- (a) In this basis the signal representations are  $c_1 = (2, 0, 0, 2)^\top$ ,  $c_2 = (0, 2, 2, 0)^\top$ ,  $c_3 = (2, 0, 2, 0)^\top$ ,  $c_4 = (0, 2, 0, 2)^\top$ .
- (b) The union bound is expressed in terms of the pairwise distances  $d_{ij}$  between the signals since

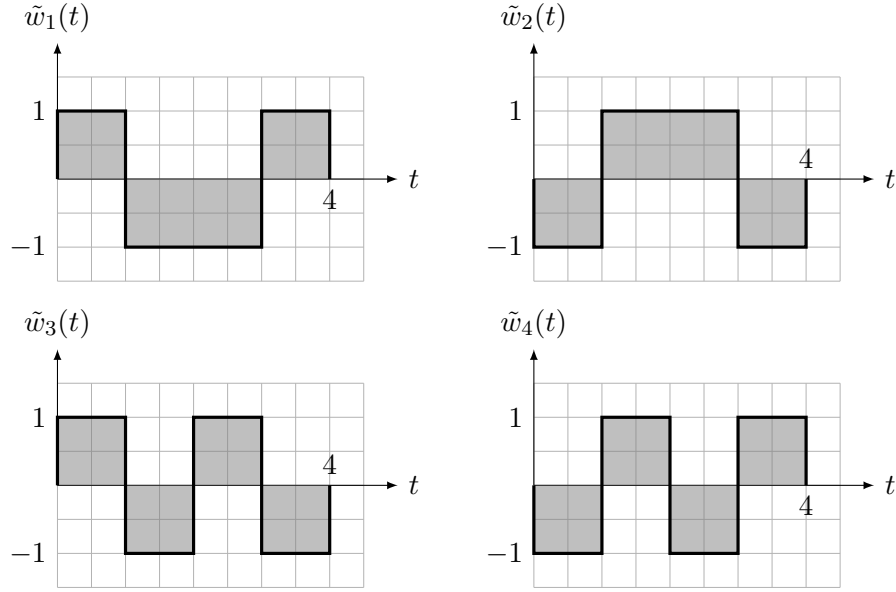
$$P_e(i) \leq \sum_{j \neq i} Q\left(\frac{d_{ij}}{2\sigma}\right)$$

From (a) we observe that  $d_{12}^2 = d_{34}^2 = 16$  and  $d_{13}^2 = d_{14}^2 = d_{23}^2 = d_{24}^2 = 8$ , hence

$$P_e(i) \leq 2Q\left(\frac{2}{\sqrt{N_0}}\right) + Q\left(\frac{2\sqrt{2}}{\sqrt{N_0}}\right)$$

Since  $P_e(i)$  does not depend on  $i$ , it also bounds the average error probability.

- (c) The minimum-energy signal set is obtained by subtracting from  $\{w_i(t)\}_{i=1}^4$  the average signal  $a(t) = \frac{1}{4} \sum_{i=1}^4 w_i(t) = \mathbb{1}_{[0,4]}(t)$ . The resulting signals are shown below.



- (d) Note that in the new signal set  $\tilde{w}_2(t) = -\tilde{w}_1(t)$  and  $\tilde{w}_4(t) = -\tilde{w}_3(t)$ . Furthermore the signals  $\tilde{w}_1(t)$  and  $\tilde{w}_3(t)$  are orthogonal. Thus the new signal space is two-dimensional, and the Gram-Schmidt procedure will produce the orthonormal basis  $\tilde{\psi}_1(t) = \frac{\tilde{w}_1(t)}{\|\tilde{w}_1\|} = \frac{1}{2}\tilde{w}_1(t)$  and  $\tilde{\psi}_2(t) = \frac{\tilde{w}_3(t)}{\|\tilde{w}_3\|} = \frac{1}{2}\tilde{w}_3(t)$ .
- (e) In the new basis the signal representations are  $\tilde{c}_1 = (2, 0)^\top$ ,  $\tilde{c}_2 = (-2, 0)^\top$ ,  $\tilde{c}_3 = (0, 2)^\top$ ,  $\tilde{c}_4 = (0, -2)^\top$ . These codewords correspond to those of the 4-QAM constellation (rotated by 45 degrees). The error probability of this set is

$$P_e = 1 - \left[ 1 - Q\left(\frac{2}{\sqrt{N_0}}\right) \right]^2 = 2Q\left(\frac{2}{\sqrt{N_0}}\right) - Q\left(\frac{2}{\sqrt{N_0}}\right)^2$$

- (f) Since translations of a signal set do not change the probability of error, the error probability of the receiver in (b) is equal to that in (e).

SOLUTION 6.

- (a) Clearly,

$$\mathcal{E}_s^C(k) = 2^{2k} - 1.$$

- (b)

$$a = Q^{-1}\left(\frac{10^{-5}}{2}\right) \approx 4.42.$$

(From the suggested approximation we get  $a \approx 4.80$ .)

- (c) For comparison, see the following table.

$k$	$\mathcal{E}_s^P(k)$	$\mathcal{E}_s^C(k)$
1	19.54	3
2	97.68	15
4	1660	255

(d) We see that

$$\frac{\mathcal{E}_s^C(k+1)}{\mathcal{E}_s^C(k)} = \frac{\mathcal{E}_s^P(k+1)}{\mathcal{E}_s^P(k)} = \frac{2^{2(k+1)} - 1}{2^{2k} - 1},$$

thus

$$\lim_{k \rightarrow \infty} \frac{\mathcal{E}_s^C(k+1)}{\mathcal{E}_s^C(k)} = \lim_{k \rightarrow \infty} \frac{\mathcal{E}_s^P(k+1)}{\mathcal{E}_s^P(k)} = 4.$$

- (e) If we send one bit per symbol, then coding allows us to significantly reduce the required energy per symbol. For every additional bit per symbol we need to multiply  $\mathcal{E}_s$  by roughly 4 (exactly 4 asymptotically) with or without coding. So as the number of bits per symbol increases, there is essentially a constant gap (in dB) between the energy per symbol required by (uncoded) PAM and that required by the best possible code.

Notice that to keep the error probability at a constant level, we need to increase  $\mathcal{E}_s/\sigma^2$  exponentially with the number  $k$  of bits per symbol. In Example 4.3 in the book we increase it linearly with  $k$  (hence the error probability goes to 1).