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Handout 17 Principles of Digital Communications
Solutions to Problem Set 7 Apr. 5, 2018

Solution 1.

(a) The Cauchy–Schwarz inequality states

$$|\langle x, y\rangle| \le \|x\| \cdot \|y\|$$

with equality if and only if $x = \alpha y$ for some scalar α. For our problem, we can write

$$|\langle w, \phi\rangle|^2 \le \|w\|^2 \cdot \|\phi\|^2 = \|w\|^2$$

with equality if and only if $\phi = \alpha w$ for some scalar α. Thus, the maximizing $\phi(t)$ is simply a scaled version of $w(t)$.

Remark. *In two dimensions, we have $|\langle x, y\rangle| = \|x\| \cdot \|y\| \cos\alpha$, where α is the angle between the two vectors. It is clear that the maximum is achieved when $\cos\alpha = 1 \Leftrightarrow \alpha = 0$ (or $\alpha = k2\pi$). Thus, x and y are colinear.*

(b) The problem is

$$\max_{\phi_1,\phi_2} (c_1\phi_1 + c_2\phi_2) \text{ subject to } \phi_1^2 + \phi_2^2 = 1$$

Thus, we can reduce by setting $\phi_2 = \sqrt{1-\phi_1^2}$ to obtain

$$\max_{\phi_1} \left(c_1\phi_1 + c_2\sqrt{1-\phi_1^2} \right)$$

This maximum is found by taking the derivative:

$$\frac{d}{d\phi_1}\left(c_1\phi_1 + c_2\sqrt{1-\phi_1^2} \right) = c_1 - c_2\frac{\phi_1}{\sqrt{1-\phi_1^2}}$$

Setting this equal to zero yields $c_1 = c_2\frac{\phi_1}{\sqrt{1-\phi_1^2}}$, i.e,

$$c_1^2 = c_2^2\frac{\phi_1^2}{1-\phi_1^2}$$

This immediately gives $\phi_1 = \frac{c_1}{\sqrt{c_1^2+c_2^2}}$ and thus $\phi_2 = \frac{c_2}{\sqrt{c_1^2+c_2^2}}$, which are colinear to c_1 and c_2 respectively.

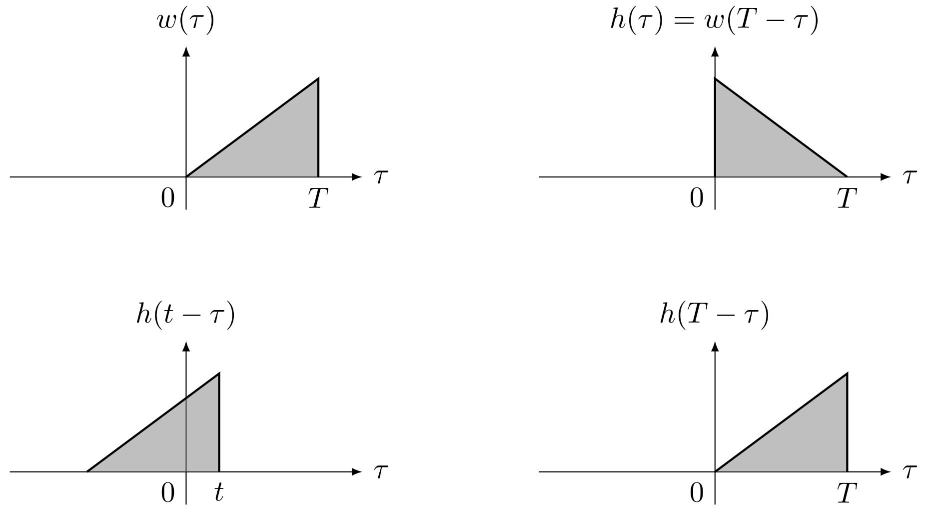
Note: the goal of this exercise was to display yet another way to derive the matched filter.

(c) Passing an input $w(t)$ through a filter with impulse response $h(t)$ generates output waveform $y(t) = \int w(\tau)h(t-\tau)d\tau$. If this waveform $y(t)$ is sampled at time $t = T$, then the output sample is

$$y(T) = \int w(\tau)h(T-\tau)d\tau \tag{1}$$

An example signal $w(\tau)$ is shown below (top left). The filter is then the waveform shown on the top right, and the convolution term of the filter on the bottom left. Finally, the filter term $h(T-\tau)$ of Equation (1) is shown on the bottom right. One can see that $h(T-\tau) = w(\tau)$, so indeed

$$y(T) = \int w(\tau)h(T-\tau)d\tau = \int w^2(\tau)d\tau = \int_0^T w^2(\tau)d\tau$$



SOLUTION 2.

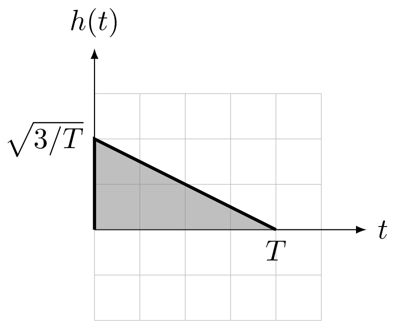
(a) The binary hypothesis testing problem may be written as:

$$\begin{aligned} H = 0: &\quad R(t) = w_1(t) + N(t) \\ H = 1: &\quad R(t) = w_2(t) + N(t) \end{aligned}$$

The impulse response of a matched filter is

$$h(t) = \frac{w_1(T-t)}{\|w_1(t)\|}$$

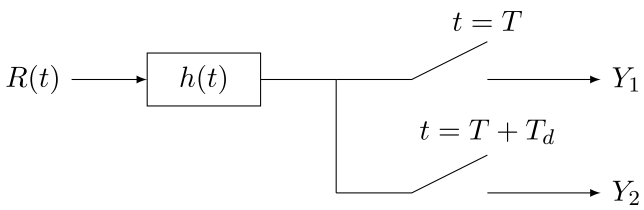
and is shown below. We have normalized the impulse response of the matched filter to have unit norm. Note that this does not affect the probability of error.



The output of the matched filter sampled at $t = T$ and $t = T + T_d$ is $Y_1 = \langle R(t), \frac{w_1(t)}{\|w_1\|} \rangle$ and $Y_2 = \langle R(t), \frac{w_2(t)}{\|w_2\|} \rangle$ respectively. The decision rule is

$$Y_1 \underset{\hat{H}=1}{\overset{\hat{H}=0}{\gtrless}} Y_2$$

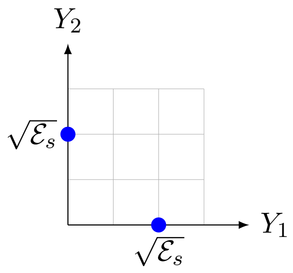
The block diagram of the system is shown below.



(b) For $T_d \geq T$, the signals $w_1(t)$ and $w_2(t)$ are orthogonal to each other. Let

$$\mathcal{E}_s = \|w_1\|^2 = \frac{A^2 T}{3}$$

(The signal space representation of the constellation can be seen below.)



The noise $Z_1, Z_2 \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$ and Z_1 is independent of Z_2. The probability of error can be readily calculated as

$$P_e = Q\left(\frac{\sqrt{2\mathcal{E}_s}}{2\sqrt{N_0/2}}\right) = Q\left(\sqrt{\frac{\mathcal{E}_s}{N_0}}\right)$$

Given that $T_d < T$, the basis functions are no longer orthogonal and the noises Z_1, Z_2 are no longer uncorrelated. But this is an excellent time to introduce the use of orthonormal basis as an isometric transformation. First, let us denote a pair of orthonormal basis functions $\tilde{w}_1(t)$, $\tilde{w}_2(t)$ and a parameter matrix $\mathbf{a}$:

$$\begin{aligned} w_1(t) &= a_{1,1}\tilde{w}_1(t) + a_{1,2}\tilde{w}_2(t) \\ w_2(t) &= a_{2,1}\tilde{w}_1(t) + a_{2,2}\tilde{w}_2(t). \end{aligned}$$

Please note that we do not need to know the actual value of $\tilde{w}_1(t)$,$\tilde{w}_2(t)$ and $\mathbf{a}$. By the usual change of basis argument, we have for every signals $f(t)$:

$$\begin{bmatrix} \langle f(t), w_1(t)\rangle \\ \langle f(t), w_2(t)\rangle \end{bmatrix} = \begin{bmatrix} a_{1,1}\langle f(t), \tilde{w}_1(t)\rangle + a_{1,2}\langle f(t), \tilde{w}_2(t)\rangle \\ a_{2,1}\langle f(t), \tilde{w}_1(t)\rangle + a_{2,2}\langle f(t), \tilde{w}_2(t)\rangle \end{bmatrix}.$$

Given the observation of our receiver in point (a) and the hypothesis H, we can apply the following linear transformation (invertible linear transformation on observed values does not change the error probability) :

$$\begin{aligned} \begin{bmatrix} \tilde{Y}_1 \\ \tilde{Y}_2 \end{bmatrix} &= \mathbf{a}^{-1}\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \\ &= \mathbf{a}^{-1}\left(\begin{bmatrix} \langle w_{H+1}(t), w_1(t)\rangle \\ \langle w_{H+1}(t), w_2(t)\rangle \end{bmatrix} + \begin{bmatrix} \langle N(t), w_1(t)\rangle \\ \langle N(t), w_2(t)\rangle \end{bmatrix}\right) \\ &= \mathbf{a}^{-1}\left(\mathbf{a}\begin{bmatrix} \langle w_{H+1}(t), \tilde{w}_1(t)\rangle \\ \langle w_{H+1}(t), \tilde{w}_2(t)\rangle \end{bmatrix} + \mathbf{a}\begin{bmatrix} \langle N(t), \tilde{w}_1(t)\rangle \\ \langle N(t), \tilde{w}_2(t)\rangle \end{bmatrix}\right) \\ &= \begin{bmatrix} \langle w_{H+1}(t), \tilde{w}_1(t)\rangle \\ \langle w_{H+1}(t), \tilde{w}_2(t)\rangle \end{bmatrix} + \begin{bmatrix} \langle N(t), \tilde{w}_1(t)\rangle \\ \langle N(t), \tilde{w}_2(t)\rangle . \end{bmatrix} \end{aligned}$$

In this new basis, the noise terms $\langle N(t), \tilde{w}_1(t)\rangle$ and $\langle N(t), \tilde{w}_2(t)\rangle$ are independent to each other with the variance of $N_0/2$. Therefore the error probability for this binary hypothesis testing problem is equal to :

$$P_e = Q\left(\frac{d}{2\sqrt{N_0/2}}\right),$$

where d is the distance between signal points under the orthonormal basis. Now we use the fact that orthonormal basis define an isometry with the waveform channel, consider a signal $f(t) = b\tilde{w}_1(t) + c\tilde{w}_2(t)$, the norm of this signal is preserved in the orthonormal basis :

$$\langle f(t), f(t)\rangle = \langle b\tilde{w}_1(t) + c\tilde{w}_2(t), b\tilde{w}_1(t) + c\tilde{w}_2(t)\rangle = b^2 + c^2 = \langle f(t), \tilde{w}_1(t)\rangle^2 + \langle f(t), \tilde{w}_2(t)\rangle^2.$$

Such that the distance d can be calculated as :

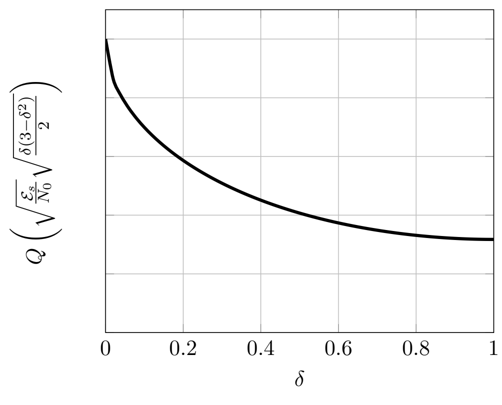
$$\begin{aligned} d^2 =& \left\| \begin{bmatrix} \langle w_1(t), \tilde{w}_1(t)\rangle \\ \langle w_1(t), \tilde{w}_2(t)\rangle \end{bmatrix} - \begin{bmatrix} \langle w_2(t), \tilde{w}_1(t)\rangle \\ \langle w_2(t), \tilde{w}_2(t)\rangle \end{bmatrix} \right\|^2 = \left\| \begin{bmatrix} \langle w_1(t) - w_2(t), \tilde{w}_1(t)\rangle \\ \langle w_1(t) - w_2(t), \tilde{w}_2(t)\rangle \end{bmatrix} \right\|^2 \\ =& ||w_1(t) - w_2(t)||^2. \end{aligned}$$

The take home message is the actual observation basis for binary hypothesis testing on waveform channel is inconsequential, as long as it can be transformed into an orthonormal basis. In our problem, the distance is equal to:

$$\begin{aligned} \|w_1(t) - w_2(t)\|^2 &= \int (w_1(t) - w_2(t))^2 dt \\ &= \int_0^{T_d} \left(\frac{A}{T}\right)^2 t^2 dt + \int_{T_d}^{T} \left(T_d \frac{A}{T}\right)^2 dt + \int_T^{T+T_d} \left(\frac{A}{T}\right)^2 (t - T_d)^2 dt \\ &= \left(\frac{A}{T}\right)^2 \left[\frac{T_d^3}{3} + T_d^2(T - T_d) + \frac{T^3 - (T - T_d)^3}{3}\right] \\ &\overset{(\star)}{=} \left(\frac{A}{T}\right)^2 \frac{1}{3} T^3 \delta(3 - \delta^2) \\ &= \mathcal{E}_s \delta(3 - \delta^2) \end{aligned}$$

where in $(\star)$ we have defined $\delta = \frac{T_d}{T}$. Given this, we can compute

$$P_e = Q\left(\sqrt{\frac{\mathcal{E}_s}{N_0}}\sqrt{\frac{\delta(3 - \delta^2)}{2}}\right)$$

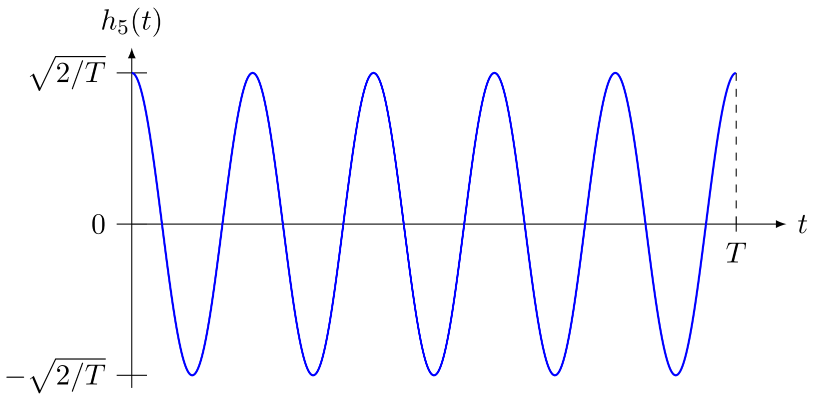


Solution 3.

(a) The matched filter is the filter whose impulse response is a delayed, time-reversed version of $w_j(t)$, i.e.

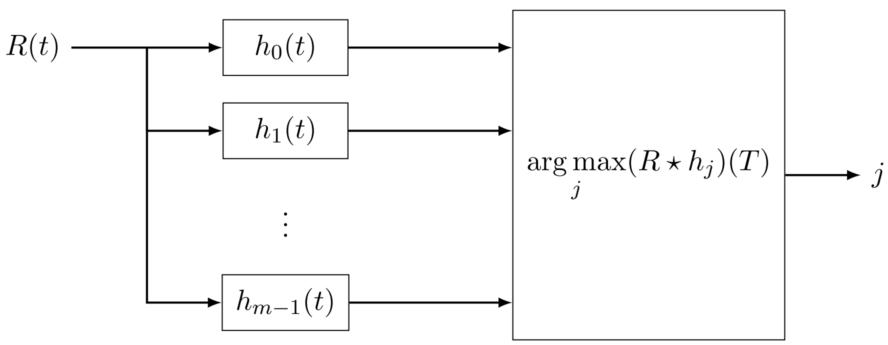
$$\begin{aligned} h_j(t) &= w_j(T-t) = \sqrt{\frac{2}{T}} \cos\left(\frac{2\pi n_j (T-t)}{T}\right) \mathbb{1}_{[0,T]}(T-t) \\ &= \sqrt{\frac{2}{T}} \cos\left(\frac{2\pi n_j t}{T}\right) \mathbb{1}_{[0,T]}(t) \end{aligned}$$

As an example, $h_5(t)$ is shown below.



The receiver then processes the received signal $R(t)$ through the matched filter $h_j(t)$ to obtain $(R \star h_j)(t)$. This signal is sampled at time T to yield the value needed for the MAP decision.

(b) We need m matched filters, one for each signal.

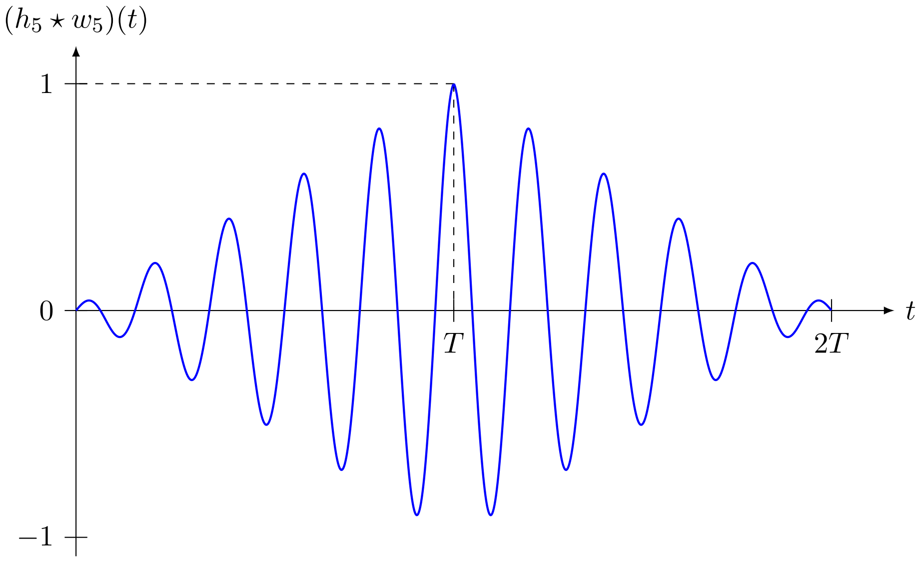


(c) The following `matlab` program computes the output of the matched filter $h_5(t)$.

```
T = 1;
Resolution = 1e-3;
t = 0:Resolution:T;
nj = 5;

wj = sqrt(2/T) * cos ( (2*pi*nj*t)/T );
hj = sqrt(2/T) * cos ( (2*pi*nj*t)/T );

output = conv(wj, hj);
```



Note that the resulting signal is *zero* for $t \leq 0$ and also for $t \geq 2T$. The figure also reveals why sampling at time $t = T$ is a good idea: the value of the matched filter output signal is maximal.

SOLUTION 4.

1. In this case all components of Y except the first will contain only WGN:

$$Y_1 = \sqrt{\mathcal{E}} + Z_1$$
$$\forall j = 2, \dots, m, \ Y_j = Z_j, \quad Z_j \sim \mathcal{N}(0, \sigma^2).$$

2. This is the event that the receiver declares $\hat{H} = 1$, since only Y_1 is larger than the threshold.

3.

$$\begin{aligned} P_e = \Pr\{(E_1 \cap E_2^c \cap E_3^c \cap \ldots \cap E_m^c)^c\} &= \Pr\{E_1^c \cup E_2 \cup E_3 \cup \ldots \cup E_m\} \\ &\leq Q\left(\frac{(1-\alpha)\sqrt{\mathcal{E}}}{\sigma}\right) + (m-1)Q\left(\frac{\alpha\sqrt{\mathcal{E}}}{\sigma}\right), \end{aligned}$$

where the inequality follows from the union bound.

4. Taking the hints given in the problem, the above expression can be written as:

$$\begin{aligned} P_e &\leq \frac{1}{2}\left(e^{-\frac{(1-\alpha)^2\mathcal{E}}{2\sigma^2}} + e^{\ln m} e^{-\frac{\alpha^2\mathcal{E}}{2\sigma^2}}\right) \\ &= \frac{1}{2}\left(e^{-\frac{(1-\alpha)^2\mathcal{E}}{2\sigma^2}} + e^{\ln m(1-\frac{\mathcal{E}_b}{2\sigma^2}\alpha^2 \log_2 e)}\right). \end{aligned}$$

The first term in the sum goes to zero as $\mathcal{E}$ grows, but the second term only diminishes if $1 - \frac{\mathcal{E}_b}{2\sigma^2}\alpha^2 \log_2 e < 0$, i.e., if

$$\frac{\mathcal{E}_b}{\sigma^2} > \frac{2\ln 2}{\alpha^2}.$$

SOLUTION 5. *(Signal translation)*

(a) Notice that

$$\|w_0(t)\|^2 = \|w_1(t)\|^2 = \int_0^{2T} w_0^2(t)dt = 2T$$

We first apply the Gram–Schmidt algorithm. We get the first basis vector from the first signal:

$$\psi_0(t) = \frac{w_0(t)}{\|w_0(t)\|} = \begin{cases} \frac{1}{\sqrt{2T}} & t \in [0, 2T] \\ 0 & \text{otherwise} \end{cases}$$

It is clear that $\psi_0(t)$ and $w_1(t)$ are orthogonal. Thus we obtain the second basis vector by normalizing $w_1(t)$:

$$\psi_1(t) = \frac{w_1(t)}{\|w_1(t)\|} = \begin{cases} \frac{1}{\sqrt{2T}} & t \in [0, T] \\ -\frac{1}{\sqrt{2T}} & t \in [T, 2T] \\ 0 & \text{otherwise} \end{cases}$$

In the $\{\psi_0(t), \psi_1(t)\}$ basis, it is straightforward to see that $c_0 = (\sqrt{2T}, 0)^\mathsf{T}$ and $c_1 = (0, \sqrt{2T})^\mathsf{T}$.

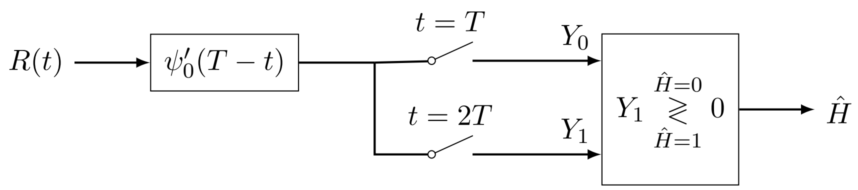
The other basis is the following:

$$\psi'_0(t) = \begin{cases} \frac{1}{\sqrt{T}} & t \in [0, T] \\ 0 & \text{otherwise} \end{cases} \qquad \psi'_1(t) = \begin{cases} \frac{1}{\sqrt{T}} & t \in [T, 2T] \\ 0 & \text{otherwise} \end{cases}$$

Observe that $\psi'_1(t) = \psi'_0(t - T)$. Hence, one matched filter at the receiver sampled twice suffices to project the received signal onto $\psi'_0(t)$ and $\psi'_1(t)$.

In the $\{\psi'_0(t), \psi'_1(t)\}$ basis, the codewords are $c_0 = (\sqrt{T}, \sqrt{T})^\mathsf{T}$ and $c_1 = (\sqrt{T}, -\sqrt{T})^\mathsf{T}$.

(b) The ML receiver is shown below.



Notice that Y_0 is not used. This is not surprising when we look at the signals: For $t \in [0, T]$, the two signals are identical.

(c) We calculate

$$\|w_0(t) - w_1(t)\| = 2\sqrt{T},$$

hence

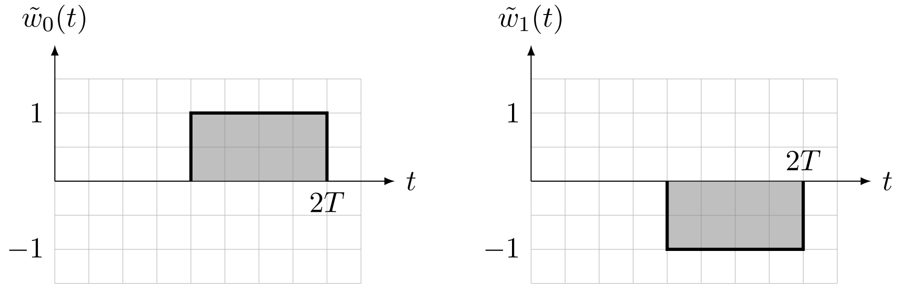
$$P_e = Q\left(\frac{\sqrt{T}}{\sqrt{N_0/2}}\right)$$

(d) Translating the signal points by any vector will not influence the error probability. However, if the translation vector is the center of mass of the original signal constellation, then the resulting signals will have minimum energy. We compute $v(t) = \frac{1}{2}w_0(t) + \frac{1}{2}w_1(t)$, thus

$$\tilde{w}_0(t) = w_0(t) - v(t) = \begin{cases} 1 & \text{for } t \in [T, 2T] \\ 0 & \text{otherwise} \end{cases}$$

$$\tilde{w}_1(t) = w_1(t) - v(t) = \begin{cases} -1 & \text{for } t \in [T, 2T] \\ 0 & \text{otherwise} \end{cases}$$

The resulting signal waveforms are shown below:



(e) The new signal constellation is antipodal. One can see that

$$\tilde{w}_0(t) = w_0(t) - v(t) = \frac{1}{2}w_0(t) - \frac{1}{2}w_1(t)$$
$$\tilde{w}_1(t) = w_1(t) - v(t) = \frac{1}{2}w_1(t) - \frac{1}{2}w_0(t) = -\tilde{w}_0(t)$$

This shows that we obtain an antipodal signal constellation regardless of the initial waveforms.

Solution 6. *(Orthogonal signal sets)*

(a) We first compute the centroid of the signal set:

$$a(t) = \sum_{j=0}^{m-1} P_H(j) w_j(t) = \frac{1}{m} \sum_{j=0}^{m-1} w_j(t)$$

The minimum-energy signal set is then obtained by translation:

$$\begin{aligned} \tilde{w}_j(t) = w_j(t) - a(t) &= w_j(t) - \frac{1}{m} \sum_{i=0}^{m-1} w_i(t) \\ &= \frac{m-1}{m} w_j(t) - \frac{1}{m} \sum_{i \neq j} w_i(t) \end{aligned}$$

(b)

$$\begin{aligned}
\|\tilde{w}_j(t)\|^2 &= \langle \tilde{w}_j(t), \tilde{w}_j(t)\rangle \\
&= \langle \frac{m-1}{m} w_j(t) - \frac{1}{m}\sum_{i\neq j} w_j(t), \frac{m-1}{m} w_j(t) - \frac{1}{m}\sum_{k\neq j} w_k(t)\rangle \\
&= \left(\frac{m-1}{m}\right)^2 \mathcal{E} + \frac{1}{m^2}\sum_{i\neq j}\sum_{k\neq j}\langle w_i(t), w_k(t)\rangle \\
&= \left(\frac{m-1}{m}\right)^2 \mathcal{E} + \frac{m-1}{m^2}\mathcal{E} = \left(1-\frac{1}{m}\right)\mathcal{E},
\end{aligned}$$

and since all signals in $\tilde{\mathcal{W}}$ are equiprobable, we obtain $\tilde{\mathcal{E}} = \left(1-\frac{1}{m}\right)\mathcal{E}$. The energy saving is therefore $\mathcal{E} - \tilde{\mathcal{E}} = \frac{1}{m}\mathcal{E}$. Alternatively, we could use $\mathcal{E} - \tilde{\mathcal{E}} = \|a(t)\|^2 = \frac{1}{m}\mathcal{E}$.

(c) Notice that $\sum_{j=0}^{m-1} \tilde{w}_j(t) = 0$ by the definition of $\tilde{w}_j(t)$, $j = 0, 1, \ldots, m-1$. Hence the m signals $\{\tilde{w}_0(t), \ldots, \tilde{w}_{m-1}(t)\}$ are linearly dependent. This means that their space has dimensionality less than m. We show that any collection of $m-1$ or less is linearly independent. That would prove that the dimensionality of the space $\{\tilde{w}_0(t), \ldots, \tilde{w}_{m-1}(t)\}$ is $m-1$. Without loss of essential generality we consider $\tilde{w}_0(t), \ldots, \tilde{w}_{m-2}(t)$. Assume that $\sum_{j=0}^{m-2} \alpha_j \tilde{w}_j(t) = 0$. Using the definition of $\tilde{w}_j(t)$, we may write

$$\sum_{j=0}^{m-2} \alpha_j \left(w_j(t) - \frac{1}{m}\sum_{i=0}^{m-1} w_i(t) \right) = 0,$$

$$\left(\sum_{j=0}^{m-2} \alpha_j w_j(t)\right) - \left(\frac{1}{m}\sum_{j=0}^{m-2}\alpha_j\right)\sum_{i=0}^{m-1} w_i(t) = 0,$$

$$\left(\sum_{j=0}^{m-2} \alpha_j w_j(t)\right) - \left(\beta\sum_{i=0}^{m-1} w_i(t)\right) = 0,$$

where $\beta = \frac{1}{m}\sum_{j=0}^{m-2}\alpha_j$. Therefore,

$$\sum_{j=0}^{m-2}(\alpha_j - \beta) w_j(t) - \beta w_{m-1}(t) = 0.$$

But $w_0(t), w_1(t), \ldots, w_{m-1}(t)$ is an orthogonal set and this implies $\beta = 0$ and $\alpha_j = \beta = 0,\ j = 0, 1, \ldots, m-2$. Hence $\tilde{w}_j(t),\ j = 0, 1, \ldots, m-2$ are linearly independent. We have proved that the new set spans a space of dimension $m-1$.