ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 13 Principles of Digital Communications

Solutions to Quiz 1 Mar. 22, 2018

SOLUTION 1.

In binary hypothesis testing, the statistic $T(\mathbf{Y})$ is a sufficient if and only if the log-likelihood ratio can be specified by $T(\mathbf{Y})$. For our problem, the log-likelihood ratio is given by

$$\log\left(\frac{p_H(1) f_{\mathbf{Y}|H}(\mathbf{y}\,|H=1)}{p_H(0) f_{\mathbf{Y}|H}(\mathbf{y}\,|H=0)}\right) = -\frac{n}{2}\log(2) + \log\left(\frac{p_H(1)}{p_H(0)}\right) + \sum_{i=1}^{n}\frac{y_i^2}{4}.$$

Therefore, for each of the proposed statistics, we need to see if it fully determines the value of the log-likelihood ratio. Going in this direction, we have:

(a) $T_1(\mathbf{Y}) = \sum_i Y_i$. (Not a sufficient statistic)

(b) $T_2(\mathbf{Y}) = \sum_i Y_i^2$. (Sufficient statistic)

(c) $T_3(\mathbf{Y}) = \sum_i |Y_i|$. (Not a sufficient statistic)

(d) $T_4(\mathbf{Y}) = \max_i |Y_i|$. (Not a sufficient statistic)

SOLUTION 2.

(a) The variable $\tilde{Y}$ is a sufficient statistic. We can perform Fisher-Neyman factorization on the output conditional probability density function :

$$\begin{aligned} f_{Y_1,Y_2|H}(y_1,y_2|i) &= \frac{1}{2\pi\sigma^2}\exp\left(-\frac{(y_1-a_i)^2+(y_2+a_i)^2}{2\sigma^2}\right) \\ &= \exp\left(\frac{a_i}{\sigma^2}(y_1-y_2-a_i)\right)\left(\frac{1}{2\pi\sigma^2}\exp\left(-\frac{1}{2\sigma^2}\left(y_1^2+y_2^2\right)\right)\right) \end{aligned}$$

Using the Fisher-Neyman Factorization Theorem, we see that

$$\begin{aligned} T(y_1,y_2) &= y_1 - y_2 \\ g_i(x) &= \exp\left(\frac{a_i}{\sigma^2}(x-a_i)\right) \\ h(y) &= \frac{1}{2\pi\sigma^2}\exp\left(-\frac{1}{2\sigma^2}\left(y_1^2+y_2^2\right)\right) \end{aligned}$$

(b) The variable $\tilde{Y}$ can also be written as,

$$\tilde{Y} = 2a_i + \underbrace{Z_1 - Z_2}_{\tilde{Z}}.$$

The noise term $\tilde{Z}$ will have a Gaussian distribution with variance $2\sigma^2$. We can compare it with the new observation given in the problem, $2U$ (note that scaling does not change the error probability),

$$2U = 2a_i + 2W$$

The noise term $2W$ is Gaussian distributed with variance $\frac{4\sigma^2}{3}$. We can see that both noise terms are Gaussian distributed with the noise term in $2U$ having smaller variance. Therefore, we should choose the new observation U.

Solution 3.

(a) The log-likelihood ratio is given by

$$\log\left(\frac{f_{Y|H}(y\,|H=1)}{f_{Y|H}(y\,|H=0)}\right) = |y-1| - |y+1|.$$

Due to assumption of equally-likely hypotheses, the following decision rule minimizes the error probability,

$$|y-1| - |y+1| \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} 0,$$

which is equivalent to

$$y \underset{\hat{H}=1}{\overset{\hat{H}=0}{\gtrless}} 0.$$

(b) For the optimal decision rule we computed in (a), the decision boundary is given by $y=0$. We can calculate the probability of error given $H=1$ as

$$P(\hat{H}=0|H=1) = \int_1^\infty \frac{1}{2}\exp(-y)\,dy = \frac{e^{-1}}{2}.$$

By symmetry we have the error probability $P_e = P(\hat{H}=0 \mid H=1) = P(\hat{H}=1 \mid H=0)$.

(c) The Bhattacharyya bound is given by

$$\begin{aligned}
P_e &\le \int_{-\infty}^{\infty} \sqrt{f_{Y|H}\left(y\,|H=0\right) f_{Y|H}\left(y\,|H=1\right)}\,dy \\
&= \frac{1}{2}\int_{-\infty}^{\infty} \exp\left(-\frac{|y-1|+|y+1|}{2}\right)\,dy.
\end{aligned}$$

The exponent can be represented as a piecewise linear function

$$-\frac{|y-1|+|y+1|}{2} = \begin{cases} y & y < -1 \\ -1 & -1 \le y \le 1 \\ -y & y > 1. \end{cases}$$

We divide the integration region according to these intervals such that we have

$$\begin{aligned}
P_e &\le \frac{1}{2}\left(\int_{-\infty}^{-1} e^y\,dy + \int_{-1}^{1} e^{-1}\,dy + \int_1^\infty e^{-y}\,dy\right) \\
&= \frac{1}{2}\left(e^{-1} + 2e^{-1} + e^{-1}\right) \\
&= 2e^{-1}.
\end{aligned}$$