Exercise Set 2 : 28 February 2019
Calcul Quantique

Exercise 1 *Matrix representations of a few gates*

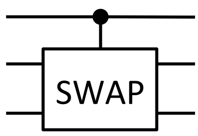
Consider the following component representation of the canonical computational basis for a quantum bit $|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$, $|1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$, and for two quantum bits $|0\rangle \otimes |0\rangle = |00\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$, $|0\rangle \otimes |1\rangle = |01\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$, $|1\rangle \otimes | = 0\rangle|10\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$, $|1\rangle \otimes |1\rangle = |11\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$.

(a) Give a matrix representation for the following reversible gates : NOT ; CNOT ; CCNOT.

(b) Recognize that these are all permutation matrices. What is their inverse matrix ?

Exercise 2 *Fredkin gate*

The SWAP operation takes two input bits and permutes them : $\text{SWAP}|b_1, b_2\rangle = |b_2, b_1\rangle$. The Fredkin gate is a three input controlled SWAP gate and is reversible. The gate swaps the two last bits if the first bit is a 1. Otherwise it leaves the input bits unchanged. One intriguing particularity of the Fredkin gate is that it conserves the number of ones.



(a) Show that the irreversible gates AND, OR can be represented in a reversible way from the Fredkin gate.

(b) Give the matrix representation of the Fredkin gate.

(c) Represent the Toffoli (CCNOT) gate in terms of {Fredkin, CNOT}.
<u>Hint</u> : You can achieve with at most one Fredkin gate and two CNOT gates.

Exercise 3 *The Mach-Zehnder interferometer.*

Consider the following marix product H(NOT)H.

(a) Is the product unitary ? Why ?

(b) Compute the output when the input is $|0\rangle$, $|1\rangle$, $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$.

(c) Draw the circuit and interpret it as a quantum interferometer.

(d) Describe the measurement outcomes at the output of the circuit (interferometer) when we measure in the computational basis.

Exercise 4 *Production of Bell states*

a) Compute the four Bell states using the following identity using Dirac's notation. Do not use the component and matrix representations.

$$|B_{xy}\rangle = (CNOT)(H \otimes I)|x\rangle \otimes |y\rangle.$$

where $x, y \in \{0, 1\}$ and $|B_{xy}\rangle$ are the Bell states.

b) Represent the corresponding circuit.

c) Represent the circuit corresponding to the inverse identity :

$$|x\rangle \otimes |y\rangle = (H \otimes I)(CNOT)|B_{xy}\rangle$$