Exercise Set 1 : 21 February 2019
Calcul Quantique

Exercise 1 *Dirac's notation for vectors and matrices*

Let $\mathcal{H} = \mathbb{C}^N$ be a vector space of N dimensional vectors with complex components. Let

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{pmatrix}$$

be a column vector. We define its "conjugate" as

$$\vec{v}^\dagger = \vec{v}^{T,*} = (v_1^*, \dots, v_N^*)$$

where * is complex conjugate. So $\vec{v}^\dagger$ is obtained by transposition and complex conjugation (if the components are real this reduces to the usual transposed vector) The inner or scalar product is by definition

$$\vec{v}^\dagger \cdot \vec{w} = v_1^* w_1 + \cdots + v_N^* w_N$$

and the norm is

$$\|\vec{v}\|^2 = \vec{v}^\dagger \cdot \vec{v} = v_1^* v_1 + \cdots + v_N^* v_N = |v_1|^2 + \cdots + |v_N|^2$$

In Dirac's notation we write $\vec{v} = |v\rangle$ and $\vec{v}^\dagger = \langle v|$. Therefore the inner product becomes

$$\langle v|w\rangle = v_1^* w_1 + \cdots + v_N^* w_N$$

The canonical orthonormal basis vectors are by definition

$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, \vec{e}_N = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

In Dirac's notation the orthonormality of the basis vectors is expressed as

$$\langle e_i|e_j\rangle = \delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j. \end{cases}$$

The expansion of a vector on this basis is

$$|v\rangle = v_1 |e_1\rangle + v_2 |e_2\rangle + \cdots + v_N |e_N\rangle$$

Now you will check a few easy facts of linear algebra and translate them in Dirac's notation.

(a) Check from the definitions above that if $|v\rangle = \alpha\,|v'\rangle + \beta\,|v''\rangle$ then

$$\langle v| = \alpha^*\,\langle v'| + \beta^*\,\langle v''|\,.$$

(b) In particular deduce that if $|v\rangle = v_1\,|e_1\rangle + v_2\,|e_2\rangle + \cdots + v_N\,|e_N\rangle$ then

$$\langle v| = v_1^*\,\langle e_1| + v_2^*\,\langle e_2| + \cdots + v_N^*\,\langle e_N|\,.$$

(c) Show directly in Dirac notation that

$$\langle v|w\rangle = v_1^* w_1 + \cdots + v_N^* w_N.$$

(d) Deduce that $\sqrt{\langle v|v\rangle} = \|v\|$.

(e) Consider the ket-bra expression $|e_i\rangle\,\langle e_j|$ for canonical basis vectors. Write this as an $N \times N$ matrix.

(f) Consider now an $N \times N$ matrix A with complex matrix elements $a_{ij}\,;\ i = 1 \ldots N\,;$ $j = 1 \ldots N$. Deduce from the above question that

$$A = \sum_{i,j=1}^{N} a_{ij}\,|e_i\rangle\,\langle e_j|$$

and also that

$$a_{ij} = \langle e_i|\,A\,|e_j\rangle\,.$$

(g) In particular verify that the identity matrix satisfies :

$$I = \sum_{i=1}^{N} |e_i\rangle\,\langle e_i|\,.$$

This is called the closure relation.

(h) (Spectral theorem) Let $A = A^\dagger$ where $A^\dagger = A^{T,*}$. This is caleld a *hermitian matrix.* An important theorem of linear algebra states that : “any hermitian matrix has N orthonormal eigenvectors with real eigenvalues (possibly degenerate)”. Let $|\varphi_i\rangle\,, \alpha_i$, $i = 1, \cdots, N$ be the eigenvectors and eigenvalues of A, *i.e.*,

$$A\,|\varphi_i\rangle = \alpha_i\,|\varphi_i\rangle\,.$$

Prove direclty in Dirac’s notation that

$$A = \sum_{i=1}^{N} \alpha_i\,|\varphi_i\rangle\,\langle\varphi_i|\,.$$

This “expansion” is often called the spectral expansion (or theorem).

Exercise 2 *Tensor Product in Dirac's notation*

Let $\mathcal{H}_1 = \mathbb{C}^N$ and $\mathcal{H}_2 = \mathbb{C}^M$ be N and M dimensional Hilbert spaces. The tensor product space $\mathcal{H}_1 \otimes \mathcal{H}_2$ is a new Hilbert space formed by "pairs of vectors" denoted as $|v\rangle_1 \otimes |w\rangle_2 \equiv |v, w\rangle$ with the properties :

- $\big(\alpha |v\rangle_1 + \beta |v'\rangle_1\big) \otimes |w\rangle_2 = \alpha |v\rangle_1 \otimes |w\rangle_2 + \beta |v'\rangle_1 \otimes |w\rangle_2$,
- $|v\rangle_1 \otimes \big(\alpha |w\rangle_2 + \beta |w'\rangle_2\big) = \alpha |v\rangle_1 \otimes |w\rangle_2 + \beta |v\rangle_1 \otimes |w'\rangle_2$,
- $\big(|v\rangle_1 \otimes |w\rangle_2\big)^\dagger = \langle v|_1 \otimes \langle w|_2$,
- $\langle v, w|v', w'\rangle = \langle v|v'\rangle_1 \langle w|w'\rangle_2$.

(a) Show that for any two vectors of $\mathcal{H}_1$ and $\mathcal{H}_2$ expanded on two basis, $|v\rangle_1 = \sum_{i=1}^N v_i |e_i\rangle_1$ and $|w\rangle_2 = \sum_{j=1}^M w_j |f_j\rangle_2$, then

$$|v\rangle_1 \otimes |w\rangle_2 = \sum_{i=1}^N \sum_{j=1}^M v_i w_j |e_i\rangle_1 \otimes |f_j\rangle_2 .$$

(b) Show that if $\{|e_i\rangle_1 ; i = 1 \dots N\}$ and $\{|f_j\rangle_2 ; j = 1 \dots M\}$ are orthonormal, then $|e_i\rangle_1 \otimes |f_j\rangle_2 \equiv |e_i, f_j\rangle$ is an orthonormal basis of $\mathcal{H}_1 \times \mathcal{H}_2$. What is the dimension of $\mathcal{H}_1 \times \mathcal{H}_2$?

(c) Any vector $|\Psi\rangle$ of $\mathcal{H}_1 \times \mathcal{H}_2$ can be expanded on the basis $|e_i\rangle_1 \otimes |f_j\rangle_2 \equiv |e_i, f_j\rangle, i = 1 \dots N, j = 1 \dots M$,

$$|\Psi\rangle = \sum_{i=1,j=1}^{N,M} \psi_{ij} |e_i, f_j\rangle .$$

If A is a matrix acting on $\mathcal{H}_1$ and B is a matrix acting on $\mathcal{H}_2$, the tensor product $A \otimes B$ is defined as

$$A \otimes B |\Psi\rangle = \sum_{i,j} \psi_{ij} A |e_i\rangle_1 \otimes B |f_j\rangle_2 .$$

Check that the matrix elements of $A \otimes B$ in the basis $|e_i, f_j\rangle$ are :

$$\langle e_i, f_j| A \otimes B |e_k, f_l\rangle = a_{ik} b_{jl}.$$

(d) Let $\mathcal{H}_1 = \mathbb{C}^2$, $\mathcal{H}_2 = \mathbb{C}^2$. Take $A_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $B_2 = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$, $|v\rangle_1 = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$, $|w\rangle_2 = \begin{pmatrix} \gamma \\ \delta \end{pmatrix}$. From the defining properties of the tensor product deduce the the following formulas :

$$|v\rangle_1 \otimes |w\rangle_2 = \begin{pmatrix} \alpha\gamma \\ \alpha\delta \\ \beta\gamma \\ \beta\delta \end{pmatrix}, \qquad A_1 \otimes B_2 = \begin{pmatrix} ae & af & be & bf \\ ag & ah & bg & bh \\ ce & cf & de & df \\ cg & ch & dg & dh \end{pmatrix}.$$

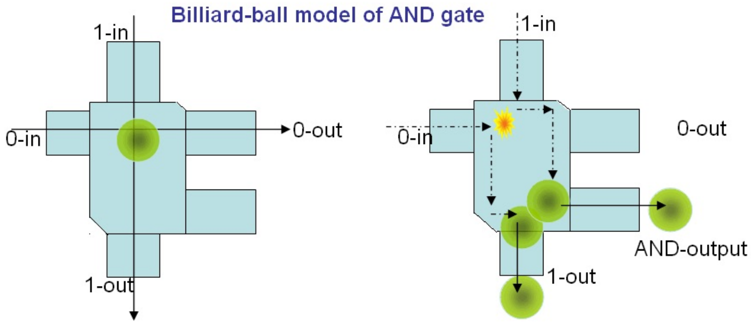
These are often useful in order to do calculations in components.

Exercise 3 *Billiard Ball Model of a classical computation (cultural aside)*

This is a physical model of computation that entirely functions through the laws of elastic collisions. It is a *reversible* and *conservative* model of computation. Here "reversible" means there is no heat disipation. And "conservative" means that the mass or equivalently the number of balls is conserved.

Here we just illustrate the flavor of this model invented by Fredkin and Toffoli by considering two gates.

(a) Convince yourself that the following operates as an AND gate. Discuss this with your fellow students.



[Figure from "Billiard-ball computer," Wikipedia, The Free Encyclopedia]

(b) Convince yourself that the following implements the Fredkin gate. Convince yourself that the following implements the Fredkin gate. The Fredkin gate is a (classical) gate that operates on three (classical) bits as a controlled SWAP. In other words FREDKIN$(b_1, b_2, b_3) = (b_1, b_2, b_3)$ if $b_1 = 0$ and FREDKIN$(b_1, b_2, b_3) = (b_1, b_3, b_2)$ if $b_1 = 1$

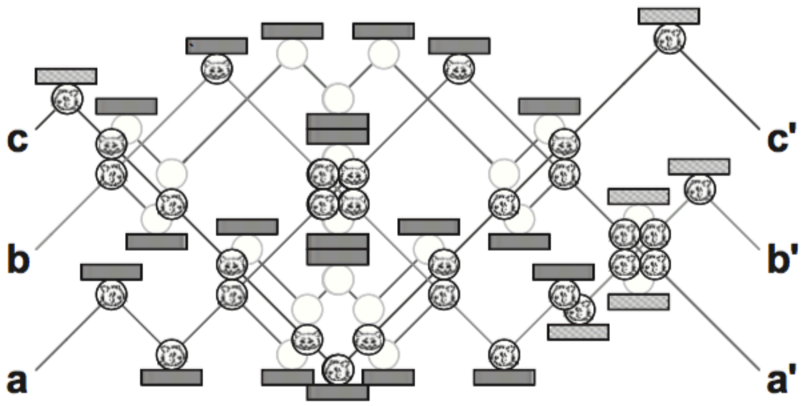


Figure 3.14. A simple billiard ball computer, with three input bits and three output bits, shown entering on the left and leaving on the right, respectively. The presence or absence of a billiard ball indicates a 1 or a 0, respectively. Empty circles illustrate potential paths due to collisions. This particular computer implements the Fredkin classical reversible logic gate, discussed in the text.

[Figure from Nielsen, Michael A., Chuang, Isaac L. (2000). Quantum Computation and Quantum Information. Cambridge, UK : Cambridge University Press.]