Ecole Polytechnique Fédérale, Lausanne: Fall 2017
September 26, 2017

# Problem Set 1 — *Due Friday, October 13, before class starts*

For the Exercise Sessions on Sept 29 and Oct 6

| Last name | First name | SCIPER Nr | Points |
|---|---|---|---|
| | | | |

**Rules :** You are allowed and encouraged to discuss these problems with your colleagues. However, each of you has to write down her solution in her own words. If you collaborated on a homework, write down the name of your collaborators and your sources. No points will be deducted for collaborations. But if we find similarities in solutions beyond the listed collaborations we will consider it as cheating. Please note that EPFL has a VERY strict policy on cheating and you might be in BIG trouble. It is simply not worth it.

**Suggested Split:** Sept 29 Problems 1–5. Oct 6 Problems 5–9.

## Problem 1: Eckart–Young Theorem

In class, we proved the converse part of the Eckart–Young theorem for the spectral norm. Here, you do the same for the case of the Frobenius norm.

*(a)* For any matrix $A$ of dimension $m \times n$ and an arbitrary orthonormal basis $\{\mathbf{x}_1, \cdots, \mathbf{x}_n\}$ of $\mathbb{C}^n$, prove that

$$\|A\|_F^2 = \sum_{k=1}^{n} \|A\mathbf{x}_k\|^2. \tag{1}$$

*(b)* Consider any $m \times n$ matrix $B$ with $\operatorname{rank}(B) \leq p$. Clearly, its null space has dimension no smaller than $n-p$. Therefore, we can find an orthonormal set $\{\mathbf{x}_1, \cdots, \mathbf{x}_{n-p}\}$ in the null space of $B$. Prove that for such vectors, we have

$$\|A - B\|_F^2 \geq \sum_{k=1}^{n-p} \|A\mathbf{x}_k\|^2. \tag{2}$$

*(c) (This requires slightly more subtle manipulations.)* For any matrix $A$ of dimension $m \times n$ and any *orthonormal* set of $n-p$ vectors in $\mathbb{C}^n$, denoted by $\{\mathbf{x}_1, \cdots, \mathbf{x}_{n-p}\}$, prove that

$$\sum_{k=1}^{n-p} \|A\mathbf{x}_k\|^2 \geq \sum_{j=p+1}^{r} \sigma_j^2. \tag{3}$$

*Hint:* Consider the case $m \geq n$ and the set of vectors $\{\mathbf{z}_1, \cdots, \mathbf{z}_{n-p}\}$, where $\mathbf{z}_k = V^H \mathbf{x}_k$. Express your formulas in terms of these and the SVD representation $A = U\Sigma V^H$.

*(d)* Briefly explain how *(a)-(c)* imply the desired statement.

**Problem 2: The Fourier matrix diagonalizes all circulant matrices.**

The discrete Fourier transform (DFT) $\mathbf{X}$ of the vector $\mathbf{x}$ is given by

$$\mathbf{X} = W\mathbf{x} \quad \text{and} \quad \mathbf{x} = \frac{1}{N} W^H \mathbf{X}. \tag{4}$$

In this homework problem, you will prove that the Fourier matrix diagonalizes all *circulant* matrices.

*(a)* To cut the derivation into two simpler steps, we introduce an auxiliary matrix $M$, defined as

$$M = WA \quad = \quad W \underbrace{\begin{pmatrix} b_0 & b_{N-1} & b_{N-2} & b_{N-3} & \dots & b_1 \\ b_1 & b_0 & b_{N-1} & b_{N-2} & \dots & b_2 \\ b_2 & b_1 & b_0 & b_{N-1} & \dots & b_3 \\ b_3 & b_2 & b_1 & b_0 & \dots & b_4 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{N-1} & b_{N-2} & b_{N-3} & b_{N-4} & \dots & b_0 \end{pmatrix}}_{\text{This is a circulant matrix}}. \tag{5}$$

Let us denote the unitary DFT of the sequence $\{b_0, b_1, \dots, b_{N-1}\}$ by $\{B_0, B_1, \dots, B_{N-1}\}$. Write out the matrix $M$ in terms of $\{B_0, B_1, \dots, B_{N-1}\}$. *Hint:* The first column of the matrix $M$ is simply given by

$$W \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_{N-1} \end{pmatrix} = \begin{pmatrix} B_0 \\ B_1 \\ B_2 \\ B_3 \\ \vdots \\ B_{N-1} \end{pmatrix} \tag{6}$$

To find the second column, you will need to use some Fourier properties.

*(b)* Using the matrix $M$ from above, compute the full matrix product

$$WAW^H \quad = \quad MW^H. \tag{7}$$

*Hint:* Handle every *row* of the matrix $M$ separately. Define the vector $\mathbf{m}$ such that $\mathbf{m}^H$ is simply the first row of the matrix $M$. But the product $\mathbf{m}^H W^H$ is easily computed, recalling that $\mathbf{m}^H W^H = (W\mathbf{m})^H$.

**Problem 3: Fourier (Review problem in view of our discussion of wavelets)**

Suppose that a signal $x(t)$ satisfies

$$\int_{-\infty}^{\infty} x(t-n)x^*(t-m)dt \quad = \quad \begin{cases} 1, & \text{if } n = m \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

(In other words, the set of functions $\{x(t-n)\}_{n\in\mathbb{Z}}$ is an orthonormal set.) Show that then, its Fourier transform $X(\omega)$ must satisfy

$$\sum_{k\in\mathbb{Z}} |X(\omega + 2\pi k)|^2 \quad = \quad 1. \tag{9}$$

**Problem 4: Hilbert Space Projection Theorem**

Given a Hilbert space $H$ and a Hilbert subspace $G$, the Hilbert space projection theorem states that for every $x \in H$, there exists a unique $y \in G$ such that

*(i)* $x - y \in G^{\perp}$

*(ii)* $\|x - y\| = \inf_{u \in G} \|x - u\|$

Recall that $G^{\perp} = \{z \in H : \langle z, x \rangle = 0 \text{ for all } x \in G\}$.

Just like in class, prove that if $y$ is indeed the minimizer of $\|x - u\|$ over all $u \in G$, then it must be true that $x - y \in G^{\perp}$, — except this time, justify every step as you "unpack" the norms into inner products, and use the properties of the inner product.

**Problem 5: Dual Basis**

In class, we have mostly discussed orthonormal bases. Let $\{\varphi_n\}_{n \in Z}$ be a basis for the Hilbert space $H$. Then, for any vector $x \in H$, we have

$$x = \sum_n \langle x, \varphi_n \rangle \varphi_n \tag{10}$$

Now, suppose that $\{\varphi'_n\}_{n \in Z}$ is also a basis for $H$, but it is *not* orthonormal. Show that if we can find a so-called *dual* basis $\{\varphi''_n\}_{n \in Z}$ satisfying $\langle \varphi'_n, \varphi''_m \rangle = \delta(n - m)$ then for any vector $x \in H$, we have

$$x = \sum_n \langle x, \varphi''_n \rangle \varphi'_n. \tag{11}$$

**Problem 6: Minimum-norm Solutions**

In this problem, we consider an *underdetermined* system of linear equations, i.e., $A\mathbf{x} = \mathbf{b}$, where $A$ is a "fat" matrix ($m < n$) and $\mathbf{b}$ is chosen such that a solution exists. As you know, in this case, there exist infinitely many solutions. Prove that the one solution $\mathbf{x}$ that has the minimum 2-norm can be expressed as

$$\mathbf{x}_{MN} = V \Sigma^{-1} U^H \mathbf{b}, \tag{12}$$

where, as usual, the SVD of $A = U \Sigma V^H$, and $\Sigma^{-1}$ is the matrix $\Sigma$ where all non-zero diagonal entries are inverted.

**Problem 7: Frames**

*(a)* We now turn to *overcomplete* expansions. The classic picture is given in Figure 1. In this picture, it is clear that every two-dimensional vector $\mathbf{x}$ can be written as

$$\mathbf{x} = a_1 \phi_1 + a_2 \phi_2 + a_3 \phi_3 \tag{13}$$

in many different ways. Explicitly and for every two-dimensional vector $\mathbf{x}$, find the solution $\mathbf{a} = (a_1, a_2, a_3)^t$ with minimum energy,[1] i.e., minimizing $a_1^2 + a_2^2 + a_3^2$. Then, give a general formula for any finite-dimensional overcomplete expansion $\{\phi_n\}_{n=1}^N$ in $k-$ dimensional space.

[1] It should also be pointed out that in some Data Science applications, we don't want the minimum-energy solution, but the *sparsest* one, i.e., the one that has the fewest non-zero coefficients. In two dimensions, this is a trivial problem, but in $N$ dimensions, there is no general simple solution, unfortunately...

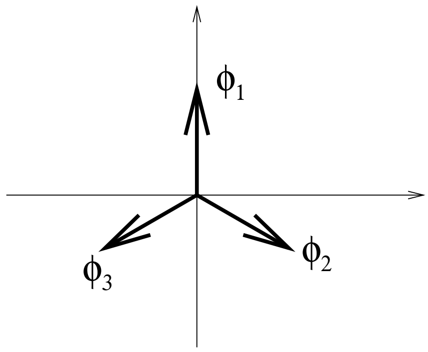


Figure 1: The three vectors $\phi_1, \phi_2$, and $\phi_3$ are at 120 degrees of each other and are of unit length each.

*(b)* There is obviously no Parseval theorem, i.e., $x_1^2 + x_2^2 \neq a_1^2 + a_2^2 + a_3^2$. An overcomplete expansion is called a *frame* if there exist constants $0 < A \leq B < \infty$ such that

$$A\|x\|^2 \leq \sum_n |\langle x, \phi_n \rangle|^2 \leq B\|x\|^2. \tag{14}$$

Find the frame bounds $A$ and $B$ for the "Mercedes" frame above. *Note:* Because frames satisfy such a Parseval-like property, they are the most common overcomplete expansions. *Another note:* If $A = B$, the frame is called *tight.*

**Problem 8: Time–Frequency Representations**

The elementary B-spline of degree 0 is the function $\beta^{(0)}(t) = 1$, for $-\frac{1}{2} \leq t < \frac{1}{2}$, and $\beta^{(0)}(t) = 0$ otherwise. The elementary B-spline of degree $K$ is defined recursively as $\beta^{(K)} = \beta^{(K-1)} * \beta^{(0)}$. Find the Heisenberg box of the elementary B-splines of orders 0 and 1 (and 2, if you like). For each case, compare the size of the Heisenberg box to the lower bound (the uncertainty principle from class).

**Problem 9: Haar Wavelet**

*This problem is taken from Vetterli/Kovacevic, p. 295.*

Consider the wavelet series expansion of continuous-time signals $f(t)$ and assume that $\psi(t)$ is the Haar wavelet.

*(a)* Give the expansion coefficients for $f(t) = 1, t \in [0, 1]$, and 0 otherwise.

*(b)* Verify that for $f(t)$ as in Part *(a),* $\sum_m \sum_n \|\langle \psi_{m,n}, f \rangle\|^2 = 1$ (i.e., Parseval's identity).

*(c)* Consider $f_1(t) = f(t - 2^{-i})$, where $i$ is a positive integer. Give the range of scales over which expansion coefficients are non-zero. (Take $f(t)$ as in Part *(a).*)

*(d)* Same as above, but now for $f_2(t) = f(t - 1/\sqrt{2})$. (Take $f(t)$ as in Part *(a).*)