Robustness, Model Uncertainty and Pricing

Antoon Pelsser[1]

[1]Maastricht University & Netspar
Email: a.pelsser@maastrichtuniversity.nl

29 October 2010
Swissquote Conference – Lausanne

Motivation

- Pricing contracts in incomplete markets
- Examples:
  - Pricing very long-dated cash flows $T \sim 30 - 100$ years
  - Pricing long-dated equity options $T > 5$ years
  - Pricing pension & insurance liabilities
- Actuarial premium principles typically "ignore" financial markets
  - Actuarial pricing is "static": price at $t = 0$ only
- Financial pricing considers "dynamic" pricing problem:
  - How does price evolve over time until time T?
- Financial pricing typically "ignores" unhedgeable risks

Main Ideas

- Pricing contracts in incomplete markets in a "market-consistent" way
- Use model uncertainty and ambiguity aversion as "umbrella"
  - Agent does not know the "true" drift rate of stochastic processes
  - Agent does know confidence interval for drift
  - Agent is worried about model mis-specification
  - Agent can trade in financial markets
  - Agent is "robust"; i.e. tries to maximise worst-case expected outcome
- Results:
  1. Robust agent perfectly hedges financial risks: leads to "risk-neutral" pricing
  2. Robust agent prices unhedgeable risks using a "worst case" drift
  3. Drift depends on type of liability: leads to non-linear pricing

Outline of This Talk

1. Literature Overview
2. Complete Market
3. Incomplete Market
4. Applications

Literature Overview

- Martingale Pricing (Föllmer-Schweizer-Schied)
  - Many possible martingale measures in incomplete market
  - Minimum variance measures
  - Quantile Hedging
- Utility Based Pricing (Carmona-book)
  - Specify utility function & find "utility indifference" price
  - Very hard problem to solve, except for special cases
  - "Horizon problem": specify utility at T
  - "Short call" problem
- Monetary Utility Functions (ADEH, Schachermayer, Filipoviç)
  - Coherent & Convex risk measures with sign-change
  - Axiomatic approach
  - Characterise as: minimum over set of "test measures" of expectation plus penalty term
  - Construct "time-consistent" risk-measures via backward induction

Literature Overview (2)

- Approaches not really different, only different "language"
- Example:
  - Minimum entropy martingale measure $\iff$
  - Exponential utility indifference price $\iff$
  - Convex risk measure with entropy penalty term
- Model Uncertainty & Robustness (Hansen-Sargent book)
  - Choose worst-case drift within "confidence interval" $\iff$
  - Coherent risk measure with given set of "test measures"
- Model Uncertainty gives economic meaning to "set of test measures"
  - Econometric estimation of parameters gives confidence intervals
  - Disagreement between panel of experts

Tree Setup

Suppose we have a stock price S with return process $x = \ln S$:

$$dx = m\,dt + \sigma\,dW_x,$$

Discretisation in binomial tree:

$$x(t+\Delta t) = x(t) + \begin{cases} +\sigma\sqrt{\Delta t} & \text{with prob. } \frac{1}{2}(1+\frac{m}{\sigma}\sqrt{\Delta t}) \\ -\sigma\sqrt{\Delta t} & \text{with prob. } \frac{1}{2}(1-\frac{m}{\sigma}\sqrt{\Delta t}). \end{cases}$$

Model uncertainty as $m \in [m_L, m_H]$. This implies that prob. in $[p_L = \frac{1}{2}(1+\frac{m_L}{\sigma}\sqrt{\Delta t}), p_H = \frac{1}{2}(1+\frac{m_H}{\sigma}\sqrt{\Delta t})]$.

Derivative Contract

Suppose we have a derivative contract with value $f\big(t+\Delta t, x(t+\Delta t)\big)$ at time $t+\Delta t$.

Taylor expansion & binomial tree:

$$f_0 = f_1 + \begin{cases} +f_x\sigma\sqrt{\Delta t} + \frac{1}{2}f_{xx}\sigma^2\Delta t & \text{with prob. } \frac{1}{2}(1+\frac{m}{\sigma}\sqrt{\Delta t}) \\ -f_x\sigma\sqrt{\Delta t} + \frac{1}{2}f_{xx}\sigma^2\Delta t & \text{with prob. } \frac{1}{2}(1-\frac{m}{\sigma}\sqrt{\Delta t}), \end{cases}$$

where $f_0 := f\big(t, x(t)\big)$, $f_1 := f\big(t+\Delta t, x(t)\big)$, $f_x := \partial f\big(t, x(t)\big)/\partial x$ and $f_{xx} := \partial^2 f\big(t, x(t)\big)/\partial x^2$.

Discounted Expectation

Rational agent calculates discounted expectation with no model uncertainty:

$$e^{-r\Delta t}\mathbb{E}_t[f\left(t+\Delta t, x(t+\Delta t)\right)] = e^{-r\Delta t}\left(f_1 + (f_x m + \tfrac{1}{2} f_{xx}\sigma^2)\Delta t\right)$$

Limit for $\Delta t \downarrow 0$ leads to pde (Feynman-Kaç formula):

$$f_t + f_x m + \tfrac{1}{2} f_{xx}\sigma^2 - rf = 0$$

Note: no "risk-neutral valuation", drift m is real-world drift.

Valuation with Model Uncertainty

Given uncertainty about drift m, "robust" rational agent will consider "worst case" discounted certainty equivalent:

$$\min_{m \in [m_L, m_H]} e^{-r\Delta t} \mathbb{E}_t^m [f(t + \Delta t, x(t + \Delta t))]$$

Explicit solution for binomial tree:

$$\begin{cases} e^{-r\Delta t}\left(f_1 + (f_x m_L + \frac{1}{2} f_{xx}\sigma^2)\Delta t\right) & \text{if } f_x > 0 \\ e^{-r\Delta t}\left(f_1 + (\qquad\quad \frac{1}{2} f_{xx}\sigma^2)\Delta t\right) & \text{if } f_x = 0 \\ e^{-r\Delta t}\left(f_1 + (f_x m_H + \frac{1}{2} f_{xx}\sigma^2)\Delta t\right) & \text{if } f_x < 0. \end{cases}$$

$\Delta t \downarrow 0$ leads to "semi-linear" pde: $f_t + f_x \overline{m} - |f_x| h + \frac{1}{2} f_{xx}\sigma^2 - rf = 0$ with $\overline{m} = \frac{1}{2}(m_H + m_L)$ and $h = \frac{1}{2}(m_H - m_L)$.

- Actuarial notion of *prudence* (not "risk-neutral")
- Coherent time-consistent risk-measure with "$\mathbb{Q} \in [p_L, p_H]$"
- Solution exists & unique: theory of BSDE's

Model Uncertainty & Hedging

Suppose that rational agent can trade in the share price S.

Buy $\theta/S(t)$ shares at t, financed by borrowing an amount θ from the bank account B.

At time $t+\Delta t$, net position has value $(e^{x(t+\Delta t)-x(t)} - e^{r\Delta t})\theta$.

Find optimal amount θ that maximises worst-case expectation:

$$\max_{\theta} \min_{m\in[m_L,m_H]} e^{-r\Delta t}\big(f_1 + (f_x m + \tfrac{1}{2} f_{xx}\sigma^2 + (m + \tfrac{1}{2}\sigma^2 - r)\theta)\Delta t\big)$$

Two-player game: "mother nature" vs. agent.

Model Uncertainty & Hedging (2)

Optimum (m, θ) depends on sign of partial deriv's:

$$\frac{\partial}{\partial \theta} : e^{-r\Delta t}(m + \tfrac{1}{2}\sigma^2 - r)\Delta t \qquad \frac{\partial}{\partial m} : e^{-r\Delta t}(f_x + \theta)\sigma \Delta t$$

Optimal choice for m depends on sign of $\frac{\partial}{\partial m}$

- Suppose agent chooses θ such that $f_x + \theta > 0$,
- then "mother nature" chooses $m = m_L$.
- $\boxed{\text{If } m_L < r - \frac{1}{2}\sigma^2}$, then agent can improve by lowering θ,
- until $\theta = -f_x$.

- Similar argument for $f_x + \theta < 0$, $\boxed{\text{if } m_H > r - \frac{1}{2}\sigma^2}$

Model Uncertainty & Hedging (3)

Conclusion: optimal choice for agent is $\theta^* = -f_x$.

- But this is delta-hedge for derivative f
- Leads to risk-neutral valuation!

How severe is restriction $m_L < r - \frac{1}{2}\sigma^2$? (Equivalent to $\mu_L < r$)

Thought-experiment:

- Suppose 25 years of data
- $\hat{\mu} = 8\%$, $\sigma = 15\%$
- Then std.err. of estimate for $\hat{\mu}$ is $\sigma/\sqrt{25} = 15\%/5 = 3\%$
- So, 95%-conf.intv. for $\hat{\mu}$ is $8\% \pm 6\%$.
- Need about $(2 * 15/(8-4))^2 \approx 50$ years of data to distinguish between 8% and 4% if $\sigma = 15\%$!

Tree Setup

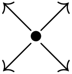
Introduce additional non-traded process y:

$$dy = a\,dt + b\,dW_y,$$

with $dW_x\,dW_y = \rho\,dt$.

"Quadrinomial" discretisation:

State:	$y + b\sqrt{\Delta t}$	$y - b\sqrt{\Delta t}$
$x + \sigma\sqrt{\Delta t}$	$p_{++} = \left(\frac{(1+\rho)+(\frac{m}{\sigma}+\frac{a}{b})\sqrt{\Delta t}}{4}\right)$	$p_{+-} = \left(\frac{(1-\rho)+(\frac{m}{\sigma}-\frac{a}{b})\sqrt{\Delta t}}{4}\right)$
$x - \sigma\sqrt{\Delta t}$	$p_{-+} = \left(\frac{(1-\rho)-(\frac{m}{\sigma}-\frac{a}{b})\sqrt{\Delta t}}{4}\right)$	$p_{--} = \left(\frac{(1+\rho)-(\frac{m}{\sigma}+\frac{a}{b})\sqrt{\Delta t}}{4}\right)$



Model Uncertainty

Model uncertainty in both m and a.

Additional notation:

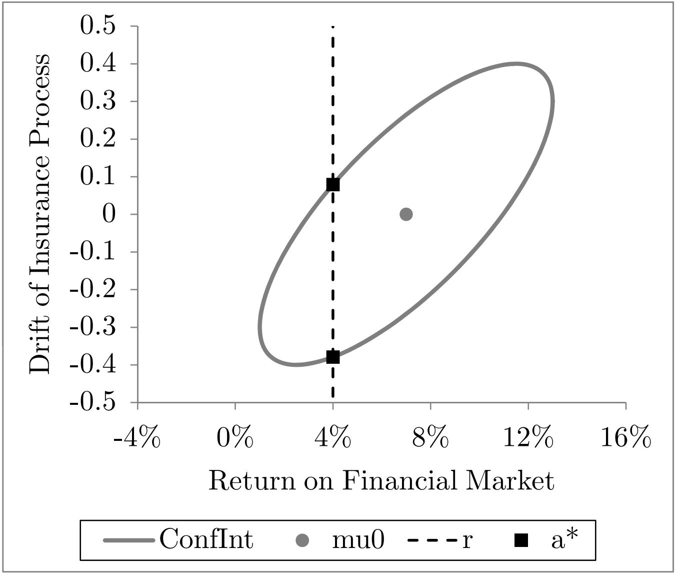
$$\mu := \begin{pmatrix} m \\ a \end{pmatrix}, \quad \Sigma := \begin{pmatrix} \sigma^2 & \rho\sigma b \\ \rho\sigma b & b^2 \end{pmatrix}.$$

Describe uncertainty set as ellipsoid:

$$\mathcal{K} := \{\mu_0 + \varepsilon \mid \varepsilon' \Sigma^{-1} \varepsilon \leq k^2\}.$$

Motivated by shape of confidence interval of estimator $\hat{\mu}$.

Ellipsoid Uncertainty Set



Robust Optimisation Problem

Robust rational agent solves the following optimisation problem

$$\max_{\theta} \min_{\mu \in \mathcal{K}} e^{-r\Delta t} \Big(f_1 + \big(f_x' \mu + \theta (e_1' \mu - r + \tfrac{1}{2}\sigma^2) + \tfrac{1}{2} \operatorname{tr}(f_{xx}\Sigma) \big) \Delta t \Big),$$

where f_x denotes gradient $(f_x, f_y)'$ and e_1 denotes the vector $(1, 0)'$.

Reformulate & simplify problem

$$\begin{aligned} \max_{\theta} \min_{\varepsilon} \quad & \theta q + \varepsilon' (f_x + \theta e_1) \\ \text{s.t.} \quad & \varepsilon' \Sigma^{-1} \varepsilon \leq k^2. \end{aligned}$$

with $q = (e_1' \mu_0 - r + \frac{1}{2}\sigma^2)$.

Optimal Response for Mother Nature

Two-player game: agent vs. "mother nature"

Worst-case choice for "mother nature" given any θ is "opposite direction" of vector $(f_x + \theta e_1)$:

$$\varepsilon^* := -\left(\frac{k}{\sqrt{(f_x + \theta e_1)'\Sigma(f_x + \theta e_1)}}\right)\Sigma(f_x + \theta e_1).$$

If we use this value for ε^* we obtain the reduced optimisation problem for the agent:

$$\max_{\theta} \quad \theta q - k\sqrt{(f_x + \theta e_1)'\Sigma(f_x + \theta e_1)}.$$

Maximise expected excess return θq minus k times st.dev. of total portfolio.

Optimal Response for Agent

Solution to reduced optimisation problem for agent:

$$\theta^* := -\left(f_x + \frac{b\rho}{\sigma} f_y\right) + \frac{q/\sigma}{\sqrt{k^2 - (q/\sigma)^2}} \frac{b\sqrt{1-\rho^2}}{\sigma} |f_y|.$$

Note, switch of notation: back to scalar expressions f_x and f_y!

Nice economic interpretation:

- Left term is "best possible" hedge
- Right term is "speculative" position, which is product of:
  - "Market confidence factor"
  - Residual unhedgeable risk

Agent's Valuation of Contract

If we substitute optimal ε^* and θ^* into original expectation, we obtain "semi-linear" pde

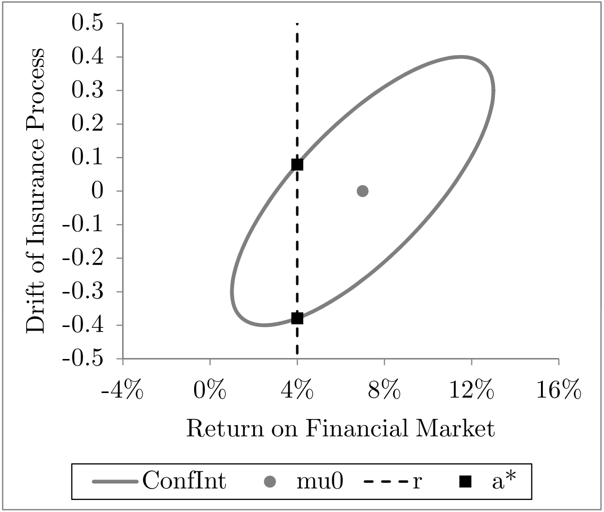
$$f_t + f_x(r - \tfrac{1}{2}\sigma^2) + f_y a^* + \tfrac{1}{2}\sigma^2 f_{xx} + \rho\sigma b f_{xy} + \tfrac{1}{2} b^2 f_{yy} - rf = 0,$$

where the drift term a^* for the insurance process is given by

$$a^* = \left(a_0 - q\frac{\rho b}{\sigma}\right) + b\sqrt{1-\rho^2} \cdot \begin{cases} \left(-\sqrt{k^2 - (q/\sigma)^2}\right) & \text{for } f_y > 0, \\ \left(+\sqrt{k^2 - (q/\sigma)^2}\right) & \text{for } f_y < 0. \end{cases}$$

Again, nice economic interpretation for a^*.

Agent's Valuation of Contract – Graphical



"Inf-convolution" of probability measures (Barrieu & El Karoui)

Generalisation to N Risk-Drivers

Suppose we have an N-dim vector x of risk-processes with covar matrix Σ and uncertainty in mean μ given by $\mathcal{K} := \{\mu_0 + \varepsilon \mid \varepsilon' \Sigma^{-1} \varepsilon \leq k^2\}$.

Suppose we can trade in $J < N$ (linear combinations of) assets. We can define a $(N \times J)$ hedge-matrix H.

Optimal hedge θ^* for agent is $\theta^* = (H'\Sigma H)^{-1} (H'\Sigma(-f_x) + \alpha H'q)$ with

$$\alpha = \sqrt{\frac{f_x' \left(\Sigma - \Sigma H(H'\Sigma H)^{-1} H'\Sigma\right) f_x}{k^2 - q' H(H'\Sigma H)^{-1} H' q}}$$

This leads to "semi-linear" pricing pde:

$$f_t + \left(r + q' \left(I - H(H'\Sigma H)^{-1} H'\Sigma\right)\right) f_x + \tfrac{1}{2}\,\mathrm{tr}(\Sigma f_{xx}) + \left(\sqrt{k^2 - q' H(H'\Sigma H)^{-1} H' q}\right) \sqrt{f_x' \left(\Sigma - \Sigma H(H'\Sigma H)^{-1} H'\Sigma\right) f_x} - rf = 0$$

Solution exists & unique: BSDE theory

Applications

- Pricing long-dated cash flows with interest rate risk.
  - N cash flows and only J bonds traded
- Pricing LT cash flow with equity & int.rate risk.
- Pricing cash flows with mortality risk.