LIBOR Market Models with Stochastic Basis

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Stylized facts

- Before the credit crunch of 2007, interest rates in the market showed typical textbook behavior. For instance:
  - A (canonical) floating rate bond is worth par at inception.
  - The forward rate implied by two deposits coincides with the corresponding FRA rate.
  - Compounding two consecutive 3m forward LIBOR rates yields the corresponding 6m forward LIBOR rate.
- These properties allowed one to construct a well-defined zero-coupon curve.
- Then August 2007 arrived, and our convictions began to waver: The liquidity crisis widened the basis between previously-near rates.
- Consider the following graphs ...

USD Market quotes

Overnight-indexed-swap rates and LIBORs

USD 3m OIS rates vs 3m Depo rates

USD Market quotes

Overnight-indexed-swap rates and LIBORs

USD 6m OIS rates vs 6m Depo rates

USD Market quotes

FRA rates and OIS forward rates

USD 3x6 FRA vs 3x6 fwd OIS

USD Market quotes

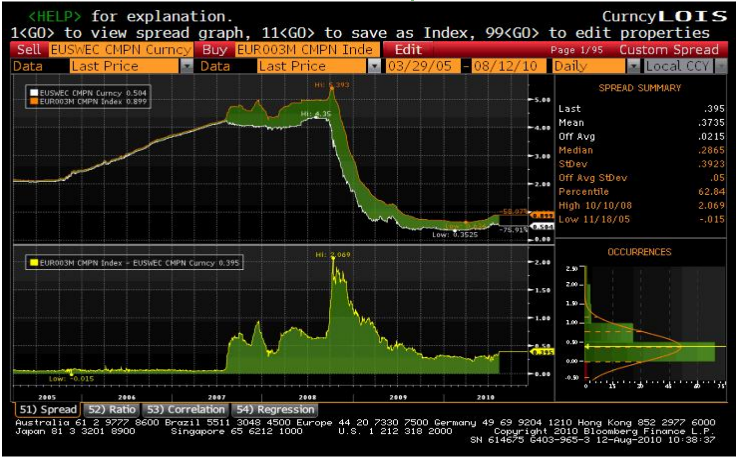
Interest rate swaps with different frequencies

USD 5y swaps: 1m vs 3m

EUR Market quotes

Overnight-indexed-swap rates and LIBORs

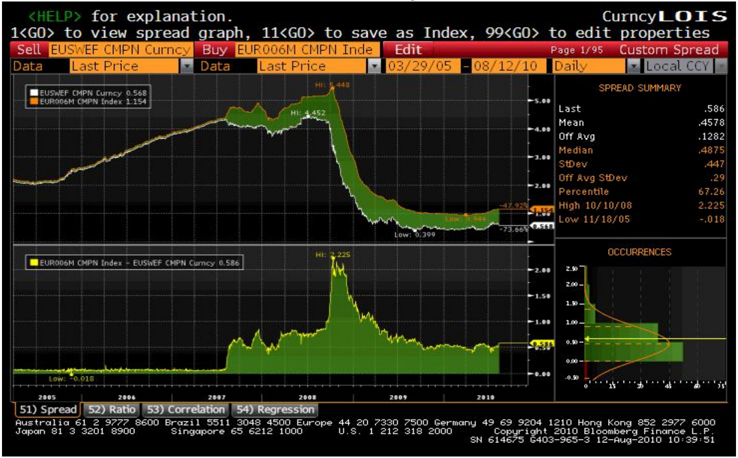
EUR 3m OIS rates vs 3m Depo rates



EUR Market quotes

Overnight-indexed-swap rates and LIBORs

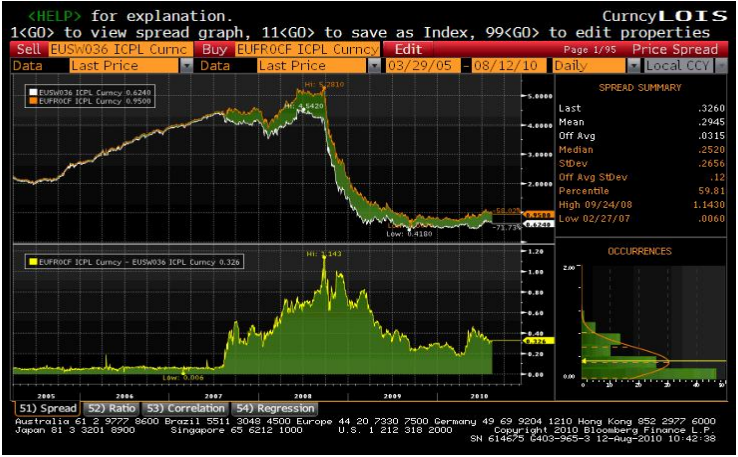
EUR 6m OIS rates vs 6m Depo rates



EUR Market quotes

FRA rates and OIS forward rates

EUR 3x6 FRA vs 3x6 fwd OIS



EUR Market quotes

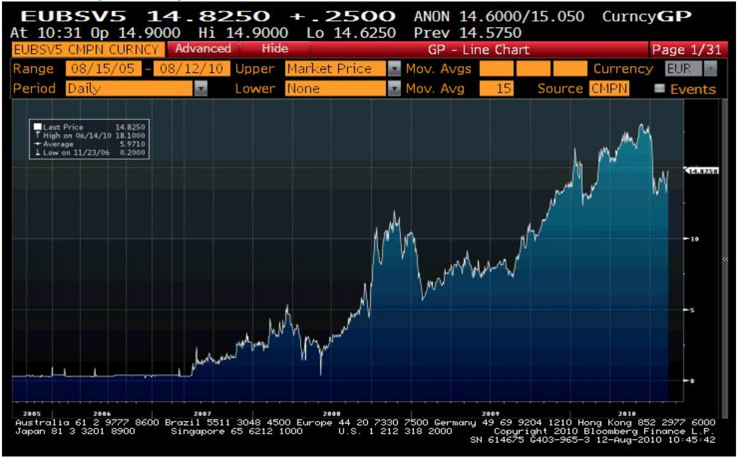
FRA rates and OIS forward rates

EUR 6x12 FRA vs 6x12 fwd OIS

EUR Market quotes

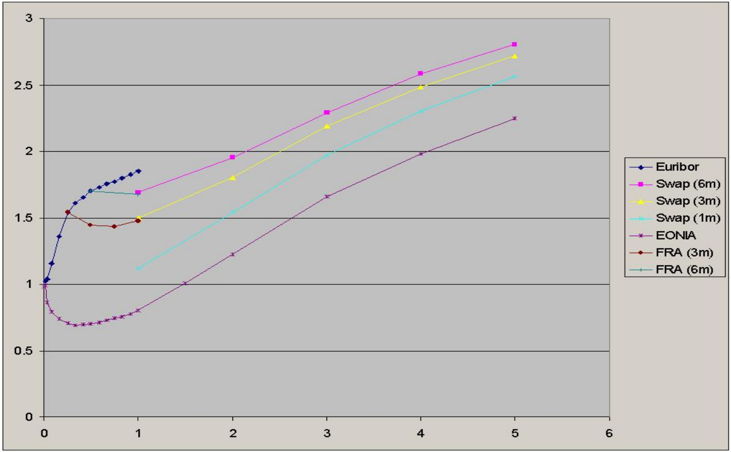
Interest rate swaps with different frequencies

EUR 5y swaps: 3m vs 6m



Market segmentation of rates

EUR market rates (as of 26 March 2009)

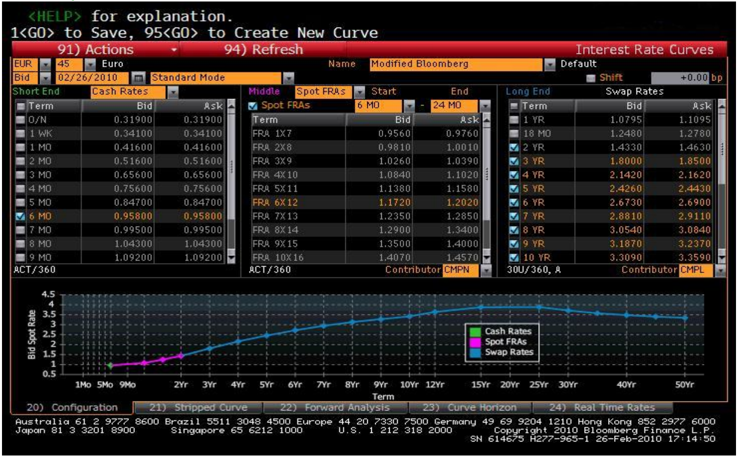


(Figure graciously provided by Kirill Levin)

Market segmentation of rates

Banks construct different zero-coupon curves for each market rate tenor: 1m, 3m, 6m, 1y, ...

Example: The EUR 6m curve



The discount curve

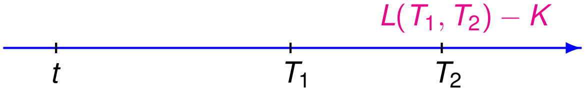
- We take the OIS zero-coupon curve, stripped from market OIS swap rates, as the discount curve:

$$T \mapsto P_D(0, T) = P^{\mathrm{OIS}}(0, T)$$

- The rationale behind this is that in the interbank derivatives market, a collateral agreement (CSA) is often negotiated between the two counterparties.
- We assume here that the collateral is revalued daily at a rate equal to the overnight rate.
- If the CSA reduces the counterparty risk to zero, it makes sense to discount with OIS rates since they can be regarded as risk-free.
- The OIS curve can be stripped from OIS swap rates using standard (single-curve) bootstrapping methods.

A new definition of forward LIBOR rate: The FRA rate

- Given times $t \leq T_1 < T_2$, the time-t FRA rate $\mathbf{FRA}(t; T_1, T_2)$ is defined as the fixed rate to be exchanged at time T_2 for the LIBOR rate $L(T_1, T_2)$ so that the swap has zero value at time t.



- Under the T_2-forward measure $Q_D^{T_2}$, we immediately have

$$\mathbf{FRA}(t; T_1, T_2) = E_D^{T_2}\left[L(T_1, T_2)|\mathcal{F}_t\right],$$

- In the classic single-curve valuation, FRA rates and corresponding discount-curve forward rates coincide:

$$\mathbf{FRA}(t; T_1, T_2) = F_D(t; T_1, T_2) = \frac{1}{T_2 - T_1}\left[\frac{P_D(t, T_1)}{P_D(t, T_2)} - 1\right]$$

A new definition of forward LIBOR rate: The FRA rate

- In fact, in the single-curve case $L(T_1, T_2)$ is defined by the classic relation

$$L(T_1, T_2) = \frac{1}{T_2 - T_1}\left[\frac{1}{P_D(T_1, T_2)} - 1\right] = F_D(T_1; T_1, T_2),$$

so that

$$\begin{aligned} &E_D^{T_2}\left[L(T_1, T_2)|\mathcal{F}_t\right] = \mathbf{FRA}(t; T_1, T_2) \\ &= E_D^{T_2}\left[F_D(T_1; T_1, T_2)|\mathcal{F}_t\right] = F_D(t; T_1, T_2) \end{aligned}$$

- In our dual-curve setting, however,

$$L(T_1, T_2) \neq F_D(T_1; T_1, T_2) = L_{OIS}(T_1, T_2)$$

implying that

$$\mathbf{FRA}(t; T_1, T_2) \neq F_D(t; T_1, T_2)$$

A new definition of forward LIBOR rate: The FRA rate

The FRA rate above is the natural generalization of a forward rate to the dual-curve case.
In fact:

1. The FRA rate coincides with the classically-defined forward rate.
2. At its reset time T_1, the FRA rate $\mathbf{FRA}(T_1; T_1, T_2)$ coincides with the LIBOR rate $L(T_1, T_2)$.
3. The FRA rate is a martingale under the corresponding forward measure.
4. Its time-0 value $\mathbf{FRA}(0; T_1, T_2)$ can be stripped from market data.

- These properties will prove to be very convenient when pricing swaps and options on LIBOR/swap rates.

The valuation of interest rate swaps (IRSs)

(under the assumption of distinct forward and discount curves)

- Given times $T_a, \ldots, T_b$, consider an IRS whose floating leg pays at each T_k the LIBOR rate with tenor $T_k - T_{k-1}$, which is set (in advance) at T_{k-1}, *i.e.*

$$\tau_k L(T_{k-1}, T_k)$$

  where τ_k denotes the year fraction.

- The time-t value of this payoff is:

$$\begin{aligned} \mathbf{FL}(t; T_{k-1}, T_k) &= \tau_k P_D(t, T_k) E_D^{T_k}\left[L(T_{k-1}, T_k)|\mathcal{F}_t\right] \\ &=: \tau_k P_D(t, T_k) L_k(t) \end{aligned}$$

  where $L_k(t) := \mathbf{FRA}(t; T_{k-1}, T_k)$.

- The swap's fixed leg is assumed to pay the fixed rate K on dates $T_c^S, \ldots, T_d^S$, with year fractions τ_j^S.

The valuation of interest rate swaps (cont'd)

- The IRS value to the fixed-rate payer is given by

$$\boxed{\mathbf{IRS}(t,K) = \sum_{k=a+1}^{b} \tau_k P_D(t,T_k) L_k(t) - K \sum_{j=c+1}^{d} \tau_j^S P_D(t,T_j^S)}$$

- We can then calculate the corresponding forward swap rate as the fixed rate K that makes the IRS value equal to zero at time t. At $t = 0$, we get:

$$\boxed{S_{0,b,0,d}(0) = \frac{\sum_{k=1}^{b} \tau_k P_D(0,T_k) L_k(0)}{\sum_{j=1}^{d} \tau_j^S P_D(0,T_j^S)}}$$

where $L_1(0)$ is the first floating payment (known at time 0).

The valuation of interest rate swaps (cont'd)

- In practice, this swap rate formula can be used to bootstrap the rates $L_k(0)$.
- The bootstrapped $L_k(0)$ can then be used to price other swaps based on the given tenor.

Swap rate	**Formulas**
OLD	$\frac{\sum_{k=1}^{b} \tau_k P(0,T_k) F_k(0)}{\sum_{j=1}^{d} \tau_j^S P(0,T_j^S)} = \frac{1-P(0,T_d^S)}{\sum_{j=1}^{d} \tau_j^S P(0,T_j^S)}$
NEW	$\frac{\sum_{k=1}^{b} \tau_k P_D(0,T_k) L_k(0)}{\sum_{j=1}^{d} \tau_j^S P_D(0,T_j^S)}$

The valuation of caplets

- Let us consider a caplet paying out at time T_k

$$\tau_k[L(T_{k-1},T_k)-K]^+.$$

- The caplet price at time t is given by:

$$\begin{aligned}\textbf{Cplt}(t,K;T_{k-1},T_k) &= \tau_k P_D(t,T_k) E_D^{T_k}\{[L(T_{k-1},T_k)-K]^+|\mathcal{F}_t\}\\ &= \tau_k P_D(t,T_k) E_D^{T_k}\{[L_k(T_{k-1})-K]^+|\mathcal{F}_t\}\end{aligned}$$

- The FRA rate $L_k(t)=E_D^{T_k}[L(T_{k-1},T_k)|\mathcal{F}_t]$ is, by definition, a martingale under $Q_D^{T_k}$.
- Assume that L_k follows a (driftless) geometric Brownian motion under $Q_D^{T_k}$.
- Straightforward calculations lead to a (modified) Black formula for caplets.

The valuation of swaptions

- A payer swaption gives the right to enter at time $T_a = T_c^S$ an IRS with payment times for the floating and fixed legs given by $T_{a+1}, \ldots, T_b$ and $T_{c+1}^S, \ldots, T_d^S$, respectively.
- Therefore, the swaption payoff at time $T_a = T_c^S$ is

$$\left[S_{a,b,c,d}(T_a) - K\right]^+ \sum_{j=c+1}^{d} \tau_j^S P_D(T_c^S, T_j^S),$$

where K is the fixed rate and

$$S_{a,b,c,d}(t) = \frac{\sum_{k=a+1}^{b} \tau_k P_D(t, T_k) L_k(t)}{\sum_{j=c+1}^{d} \tau_j^S P_D(t, T_j^S)}$$

- Assume that $S_{a,b,c,d}$ is a (lognormal) martingale under the associated swap measure.
- We thus obtain a (modified) Black formula for swaptions.

The new market formulas for caps and swaptions

Type	Formulas
OLD **Cplt**	$\tau_k P(t, T_k)\,\mathrm{Bl}\big(K, F_k(t), v_k\sqrt{T_{k-1} - t}\big)$
NEW **Cplt**	$\tau_k P_D(t, T_k)\,\mathrm{Bl}\big(K, L_k(t), \bar{v}_k\sqrt{T_{k-1} - t}\big)$
OLD **PS**	$\sum_{j=c+1}^{d} \tau_j^S P(t, T_j^S)\,\mathrm{Bl}\big(K, S_{OLD}(t), \nu\sqrt{T_a - t}\big)$
NEW **PS**	$\sum_{j=c+1}^{d} \tau_j^S P_D(t, T_j^S)\,\mathrm{Bl}\big(K, S_{a,b,c,d}(t), \bar{\nu}\sqrt{T_a - t}\big)$

The multi-curve LIBOR Market Model (LMM)

- In the classic (single-curve) LMM, one models the joint evolution of a set of consecutive forward LIBOR rates.
- What about our multi-curve case?
- When pricing a payoff depending on same-tenor LIBOR rates, it is convenient to model the FRA rates L_k.
- This choice is also convenient in the case of a swap-rate dependent payoff. In fact, we can write:

$$S_{a,b,c,d}(t) = \frac{\sum_{k=a+1}^{b} \tau_k P_D(t, T_k) L_k(t)}{\sum_{j=c+1}^{d} \tau_j^S P_D(t, T_j^S)} = \sum_{k=a+1}^{b} \omega_k(t) L_k(t)$$

$$\omega_k(t) := \frac{\tau_k P_D(t, T_k)}{\sum_{j=c+1}^{d} \tau_j^S P_D(t, T_j^S)}$$

- But, is the modeling of FRA rates enough?

The multi-curve LIBOR Market Model (LMM)

Alternative formulations

- In fact, we also need to model the OIS forward rates, $k = 1, \ldots, M$:

$$F_k(t) := F_D(t; T_{k-1}, T_k) = \frac{1}{\tau_k}\left[\frac{P_D(t, T_{k-1})}{P_D(t, T_k)} - 1\right]$$

- Denote by $S_k(t)$ the spread

$$S_k(t) := L_k(t) - F_k(t)$$

- By definition, both L_k and F_k are martingales under the forward measure $Q_D^{T_k}$, and thus their difference S_k is as well.
- The LMM can be extended to the multi-curve case in three different ways by:
  1. Modeling the joint evolution of rates L_k and F_k.
  2. Modeling the joint evolution of rates L_k and spreads S_k.
  3. Modeling the joint evolution of rates F_k and spreads S_k.

A first McLMM

Modeling rates F_k and L_k

- Let us consider a set of times $\mathcal{T} = \{0 < T_0, \dots, T_M\}$ compatible with a given tenor.
- We assume that each rate $L_k(t)$ evolves under $Q_D^{T_k}$ according to

$$\mathrm{d}L_k(t) = \sigma_k(t) L_k(t)\,\mathrm{d}Z_k(t), \quad t \leq T_{k-1}$$

- Likewise, we assume that

$$\mathrm{d}F_k(t) = \sigma_k^D(t) F_k(t)\,\mathrm{d}Z_k^D(t), \quad t \leq T_{k-1}$$

- The drift of $X \in \{L_k, F_k\}$ under $Q_D^{T_j}$ is equal to

$$\mathbf{Drift}(X; Q_D^{T_j}) = -\frac{\mathrm{d}\langle X, \ln(P_D(\cdot, T_k)/P_D(\cdot, T_j))\rangle_t}{\mathrm{d}t},$$

A first McLMM

Dynamics under a general forward measure

Proposition. The dynamics of L_k and F_k under $Q_D^{T_j}$ are:

$$j < k : \begin{cases} dL_k(t) = \sigma_k(t)L_k(t)\left[\displaystyle\sum_{h=j+1}^{k} \frac{\rho_{k,h}^{L,F}\tau_h^D\sigma_h^D(t)F_h(t)}{1+\tau_h^D F_h(t)}\,dt + dZ_k^j(t)\right] \\ dF_k(t) = \sigma_k^D(t)F_k(t)\left[\displaystyle\sum_{h=j+1}^{k} \frac{\rho_{k,h}^{D,D}\tau_h^D\sigma_h^D(t)F_h(t)}{1+\tau_h^D F_h(t)}\,dt + dZ_k^{j,D}(t)\right] \end{cases}$$

$$j = k : \begin{cases} dL_k(t) = \sigma_k(t)L_k(t)\,dZ_k^j(t) \\ dF_k(t) = \sigma_k^D(t)F_k(t)\,dZ_k^{j,D}(t) \end{cases}$$

$$j > k : \begin{cases} dL_k(t) = \sigma_k(t)L_k(t)\left[-\displaystyle\sum_{h=k+1}^{j} \frac{\rho_{k,h}^{L,F}\tau_h^D\sigma_h^D(t)F_h(t)}{1+\tau_h^D F_h(t)}\,dt + dZ_k^j(t)\right] \\ dF_k(t) = \sigma_k^D(t)F_k(t)\left[-\displaystyle\sum_{h=k+1}^{j} \frac{\rho_{k,h}^{D,D}\tau_h^D\sigma_h^D(t)F_h(t)}{1+\tau_h^D F_h(t)}\,dt + dZ_k^{j,D}(t)\right] \end{cases}$$

A first McLMM

Dynamics under the spot LIBOR measure

- The spot LIBOR measure $Q_D^{\mathcal{T}}$ associated to times $\mathcal{T} = \{T_0, \dots, T_M\}$ is the measure whose numeraire is

$$B_D^{\mathcal{T}}(t) = \frac{P_D(t, T_{\beta(t)-1})}{\prod_{j=0}^{\beta(t)-1} P_D(T_{j-1}, T_j)},$$

  where $\beta(t) = m$ if $T_{m-2} < t \le T_{m-1}$, $m \ge 1$.
- Application of the change-of numeraire technique leads to:

$$\mathrm{d}L_k(t) = \sigma_k(t) L_k(t) \left[\sum_{h=\beta(t)}^{k} \frac{\rho_{k,h}^{L,F} \tau_h^D \sigma_h^D(t) F_h(t)}{1 + \tau_h^D F_h(t)} \mathrm{d}t + \mathrm{d}Z_k^d(t) \right]$$

$$\mathrm{d}F_k(t) = \sigma_k^D(t) F_k(t) \left[\sum_{h=\beta(t)}^{k} \frac{\rho_{k,h}^{D,D} \tau_h^D \sigma_h^D(t) F_h(t)}{1 + \tau_h^D F_h(t)} \mathrm{d}t + \mathrm{d}Z_k^{d,D}(t) \right]$$

A first McLMM

The pricing of caplets

- The pricing of caplets in our multi-curve lognormal LMM is straightforward. We get:

$$\textbf{Cplt}(t, K; T_{k-1}, T_k) = \tau_k P_D(t, T_k)\,\mathrm{Bl}(K, L_k(t), v_k(t))$$

where

$$v_k(t) := \sqrt{\int_t^{T_{k-1}} \sigma_k(u)^2\,du}$$

- As expected, this formula is analogous to that obtained in the single-curve lognormal LMM.
- Here, we just have to replace the “old” forward rates with the corresponding FRA rates and use the discount factors of the OIS curve.

A first McLMM

The pricing of swaptions

- Our objective is to derive an analytical approximation for the implied volatility of swaptions.
- To this end, we recall that

$$S_{a,b,c,d}(t) = \sum_{k=a+1}^{b} \omega_k(t) L_k(t), \quad \omega_k(t) = \frac{\tau_k P_D(t, T_k)}{\sum_{j=c+1}^{d} \tau_j^S P_D(t, T_j^S)}$$

- Contrary to the single-curve case, the weights are not functions of the FRA rates only, since they also depend on discount factors calculated on the OIS curve.
- Therefore we can not write, under $Q_D^{c,d}$,

$$\mathrm{d}S_{a,b,c,d}(t) = \sum_{k=a+1}^{b} \frac{\partial S_{a,b,c,d}(t)}{\partial L_k(t)} \sigma_k(t) L_k(t) \, \mathrm{d}Z_k^{c,d}(t).$$

A first McLMM

The pricing of swaptions

- However, we can resort to a standard approximation technique and freeze weights ω_k at their time-0 value:

$$S_{a,b,c,d}(t) \approx \sum_{k=a+1}^{b} \omega_k(0) L_k(t),$$

thus also freezing the dependence of $S_{a,b,c,d}$ on rates F_h^D.

- Hence, we can write:

$$\mathrm{d}S_{a,b,c,d}(t) \approx \sum_{k=a+1}^{b} \omega_k(0)\sigma_k(t)L_k(t)\,\mathrm{d}Z_k^{c,d}(t).$$

- Like in the classic single-curve LMM, we then:
  - Match instantaneous quadratic variations
  - Freeze FRA and swap rates at their time-0 value

A first McLMM

The pricing of swaptions

- This immediately leads to the following (payer) swaption price at time 0:

$$\mathbf{PS}(0, K; T_{a+1}, \ldots, T_b, T_{c+1}^S, \ldots, T_d^S) = \sum_{j=c+1}^{d} \tau_j^S P_D(0, T_j^S)\,\mathrm{Bl}\big(K, S_{a,b,c,d}(0), V_{a,b,c,d}\big),$$

where the swaption volatility (multiplied by $\sqrt{T_a}$) is given by

$$V_{a,b,c,d} = \sqrt{\sum_{h,k=a+1}^{b} \frac{\omega_h(0)\omega_k(0)L_h(0)L_k(0)\rho_{h,k}}{(S_{a,b,c,d}(0))^2} \int_0^{T_a} \sigma_h(t)\sigma_k(t)\,\mathrm{d}t}$$

- Again, this formula is analogous in structure to that obtained in the single-curve lognormal LMM.

A second McLMM

A general framework for the single-tenor case

- Let us fix a given tenor x and consider a time structure $\mathcal{T} = \{0 < T_0^x, \ldots, T_{M_x}^x\}$ compatible with x.
- Let us define forward OIS rates by

$$F_k^x(t) := F_D(t; T_{k-1}^x, T_k^x) = \frac{1}{\tau_k^x}\left[\frac{P_D(t, T_{k-1}^x)}{P_D(t, T_k^x)} - 1\right], \;\; k = 1, \ldots, M_x,$$

where τ_k^x is the year fraction for the interval $(T_{k-1}^x, T_k^x]$, and basis spreads by

$$S_k^x(t) := L_k^x(t) - F_k^x(t), \;\; k = 1, \ldots, M_x.$$

- By definition, both L_k^x and F_k^x are martingales under the forward measure $Q_D^{T_k^x}$.
- Hence, their difference S_k^x is a $Q_D^{T_k^x}$-martingale as well.

A second McLMM

A general framework for the single-tenor case

- We define the joint evolution of rates F_k and spreads S_k under the spot LIBOR measure $Q_D^{\mathcal{T}}$, whose numeraire is

$$B_D^{\mathcal{T}}(t) = \frac{P_D(t, T^x_{\beta(t)-1})}{\prod_{j=0}^{\beta(t)-1} P_D(T^x_{j-1}, T^x_j)},$$

where $\beta(t) = m$ if $T^x_{m-2} < t \leq T^x_{m-1}$, $m \geq 1$, and $T^x_{-1} := 0$.

- Our single-tenor framework is based on assuming that, under $Q_D^{\mathcal{T}}$, OIS rates follow general SLV processes:

$$\begin{aligned} dF_k^x(t) &= \phi_k^F(t, F_k^x(t))\psi_k^F(V^F(t)) \\ &\quad \cdot \left[\sum_{h=\beta(t)}^{k} \frac{\tau_h^x \rho_{h,k} \phi_h^F(t, F_h^x(t))\psi_h^F(V^F(t))}{1 + \tau_h^x F_h^x(t)} dt + dZ_k^{\mathcal{T}}(t) \right] \\ dV^F(t) &= a^F(t, V^F(t))\, dt + b^F(t, V^F(t))\, dW^{\mathcal{T}}(t) \end{aligned}$$

A second McLMM

A general framework for the single-tenor case

where

- ϕ_k^F, ψ_k^F, a^F and b^F are deterministic functions of their respective arguments
- $Z^{\mathcal{T}} = \{Z_1^{\mathcal{T}}, \dots, Z_{M_x}^{\mathcal{T}}\}$ is an M_x-dimensional $Q_D^{\mathcal{T}}$-Brownian motion with instantaneous correlation matrix $(\rho_{k,j})_{k,j=1,\dots,M_x}$
- $W^{\mathcal{T}}$ is a $Q_D^{\mathcal{T}}$-Brownian motion whose instantaneous correlation with $Z_k^{\mathcal{T}}$ is denoted by ρ_k^x for each k.
- The stochastic volatility V^F is assumed to be a process common to all OIS forward rates.
- We assume that $V^F(0) = 1$.

Generalizations can be considered where each rate F_k^x has a different volatility process.

A second McLMM

A general framework for the single-tenor case

- We then assume that also the spreads S_k^x follow SLV processes.
- For computational convenience, we assume that spreads and their volatilities are independent of OIS rates.
- This implies that each S_k^x is a $Q_D^{\mathcal{T}}$-martingale as well.
- Finally, the global correlation matrix that includes all cross correlations is assumed to be positive semidefinite.

Remark. Several are the examples of dynamics that can be considered. Obvious choices include combinations (and permutations) of geometric Brownian motions and of the stochastic-volatility models of Hagan *et al.* (2002) and Heston (1993). However, the discussion that follows is rather general and requires no dynamics specification.

A second McLMM

Caplet pricing

- Let us consider the x-tenor caplet paying out at time T_k^x

$$\tau_k^x [L_k^x(T_{k-1}^x) - K]^+$$

- Our assumptions on the discount curve imply that the caplet price at time t is given by

$$\begin{aligned}&\mathbf{Cplt}(t, K; T_{k-1}^x, T_k^x)\\ &= \tau_k^x P_D(t, T_k^x) E_D^{T_k^x}\{[L_k^x(T_{k-1}^x) - K]^+ | \mathcal{F}_t\}\\ &= \tau_k^x P_D(t, T_k^x) E_D^{T_k^x}\{[F_k^x(T_{k-1}^x) + S_k^x(T_{k-1}^x) - K]^+ | \mathcal{F}_t\}\end{aligned}$$

- Assume we explicitly know the $Q_D^{T_k^x}$-densities $f_{S_k^x(T_{k-1}^x)}$ and $f_{F_k^x(T_{k-1}^x)}$ (conditional on $\mathcal{F}_t$) of $S_k^x(T_{k-1}^x)$ and $F_k^x(T_{k-1}^x)$, respectively, and/or the associated caplet prices.

A second McLMM

Caplet pricing

- Thanks to the independence of the random variables $F_k^x(T_{k-1}^x)$ and $S_k^x(T_{k-1}^x)$ we equivalently have:

$$\frac{\mathbf{Cplt}(t, K; T_{k-1}^x, T_k^x)}{\tau_k^x P_D(t, T_k^x)}$$
$$= \int_{-\infty}^{+\infty} E_D^{T_k^x}\left\{[F_k^x(T_{k-1}^x) - (K - z)]^+ | \mathcal{F}_t\right\} f_{S_k^x(T_{k-1}^x)}(z)\,\mathrm{d}z$$
$$= \int_{-\infty}^{+\infty} E_D^{T_k^x}\left\{[S_k^x(T_{k-1}^x) - (K - z)]^+ | \mathcal{F}_t\right\} f_{F_k^x(T_{k-1}^x)}(z)\,\mathrm{d}z$$

- One may use the first or the second formula depending on the chosen dynamics for F_k^x and S_k^x.
- To calculate the caplet price one needs to derive the dynamics of F_k^x and V^F under the forward measure $Q_D^{T_k^x}$.
- Notice that the $Q_D^{T_k^x}$-dynamics of S_k^x and its volatility are the same as those under $Q_D^{\mathcal{T}}$.

A second McLMM

Caplet pricing

- The dynamics of F_k^x and V^F under $Q_D^{T_k^x}$ are given by:

$$\begin{aligned} dF_k^x(t) &= \phi_k^F(t, F_k^x(t))\psi_k^F(V^F(t))\, dZ_k^k(t) \\ dV^F(t) &= a^F(t, V^F(t))\, dt + b^F(t, V^F(t)) \\ &\qquad \cdot \left[- \sum_{h=\beta(t)}^{k} \frac{\tau_h^x \phi_h^F(t, F_h^x(t))\psi_h^F(V^F(t))\rho_h^x}{1 + \tau_h^x F_h^x(t)}\, dt + dW^k(t) \right] \end{aligned}$$

  where Z_k^k and W^k are $Q_D^{T_k^x}$-Brownian motions.
- By resorting to standard drift-freezing techniques, one can find tractable approximations of V^F for typical choices of a^F and b^F, which will lead either to an explicit density $f_{F_k^x(T_{k-1}^x)}$ or to an explicit option pricing formula (on F_k^x).
- This, along with the assumed tractability of S_k^x, will finally allow the calculation of the caplet price.

A second McLMM

Swaption pricing

- Let us consider a (payer) swaption, which gives the right to enter at time $T_a^x = T_c^S$ an interest-rate swap with payment times for the floating and fixed legs given by $T_{a+1}^x, \dots, T_b^x$ and $T_{c+1}^S, \dots, T_d^S$, respectively, with $T_b^x = T_d^S$ and where the fixed rate is K.
- The swaption payoff at time $T_a^x = T_c^S$ is given by

$$\left[S_{a,b,c,d}(T_a^x) - K\right]^+ \sum_{j=c+1}^{d} \tau_j^S P_D(T_c^S, T_j^S),$$

where the forward swap rate $S_{a,b,c,d}(t)$ is given by

$$S_{a,b,c,d}(t) = \frac{\sum_{k=a+1}^{b} \tau_k^x P_D(t, T_k^x) L_k^x(t)}{\sum_{j=c+1}^{d} \tau_j^S P_D(t, T_j^S)}.$$

A second McLMM

Swaption pricing

- The swaption payoff is conveniently priced under $Q_D^{c,d}$:

$$\mathbf{PS}(t, K; T_a^x, \ldots, T_b^x, T_{c+1}^S, \ldots, T_d^S)$$
$$= \sum_{j=c+1}^{d} \tau_j^S P_D(t, T_j^S)\, E_D^{c,d}\left\{ \left[S_{a,b,c,d}(T_a^x) - K\right]^+ |\mathcal{F}_t\right\}$$

- To calculate the last expectation, we set

$$\omega_k(t) := \frac{\tau_k^x P_D(t, T_k^x)}{\sum_{j=c+1}^{d} \tau_j^S P_D(t, T_j^S)}$$

and write:

$$S_{a,b,c,d}(t) = \sum_{k=a+1}^{b} \omega_k(t) L_k^x(t)$$
$$= \sum_{k=a+1}^{b} \omega_k(t) F_k^x(t) + \sum_{k=a+1}^{b} \omega_k(t) S_k^x(t) =: \bar{F}(t) + \bar{S}(t)$$

A second McLMM

Swaption pricing

- The processes $S_{a,b,c,d}$, $\bar{F}$ and $\bar{S}$ are all $Q_D^{c,d}$-martingales.
- $\bar{F}$ is equal to the classic single-curve forward swap rate that is defined by OIS discount factors, and whose reset and payment times are given by $T_c^S, \ldots, T_d^S$.
- If the dynamics of rates F_k^x are sufficiently tractable, we can approximate $\bar{F}(t)$ by a driftless stochastic-volatility process, $\tilde{F}(t)$, of the same type as that of F_k^x.
- The process $\bar{S}$ is more complex, since it explicitly depends both on OIS discount factors and on basis spreads.
- However, we can resort to a standard approximation and freeze the weights ω_k at their time-0 value, thus removing the dependence of $\bar{S}$ on OIS discount factors.

A second McLMM

Swaption pricing

- We then assume we can further approximate $\bar{S}$ with a dynamics $\tilde{S}$ similar to that of S_k^x, for instance by matching instantaneous variations.
- After the approximations just described, the swaption price becomes

$$\mathbf{PS}(t, K; T_a^x, \dots, T_b^x, T_{c+1}^S, \dots, T_d^S)$$
$$= \sum_{j=c+1}^{d} \tau_j^S P_D(t, T_j^S)\, E_D^{c,d}\{[\tilde{F}(T_a^x) + \tilde{S}(T_a^x) - K]^+ |\mathcal{F}_t\}$$

which can be calculated exactly in the same fashion as the previous caplet price.

A second McLMM

A tractable class of multi-tenor models

- Let us consider a time structure $\mathcal{T} = \{0 < T_0, \dots, T_M\}$ and tenors $x_1 < x_2 < \cdots < x_n$ with associated time structures $\mathcal{T}^{x_i} = \{0 < T_0^{x_i}, \dots, T_{M_{x_i}}^{x_i}\}$.
- We assume that each x_i is a multiple of the preceding tenor x_{i-1}, and that $\mathcal{T}^{x_n} \subset \mathcal{T}^{x_{n-1}} \subset \cdots \subset \mathcal{T}^{x_1} = \mathcal{T}$.
- Forward OIS rates are defined, for each tenor $x \in \{x_1, \dots, x_n\}$, by

$$F_k^x(t) := F_D(t; T_{k-1}^x, T_k^x) = \frac{1}{\tau_k^x}\left[\frac{P_D(t, T_{k-1}^x)}{P_D(t, T_k^x)} - 1\right], \quad k = 1, \dots, M_x,$$

and basis spreads are defined by

$$S_k^x(t) = \mathbf{FRA}(t, T_{k-1}^x, T_k^x) - F_k^x(t) = L_k^x(t) - F_k^x(t), \quad k = 1, \dots, M_x.$$

- L_k^x, F_k^x, S_k^x are martingales under the forward measure $Q_D^{T_k^x}$.

A second McLMM

A tractable class of multi-tenor models

- We assume that, under the spot LIBOR measure $Q_D^{\mathcal{T}}$, the OIS forward rates $F_k^{x_1}$, $k = 1, \ldots, M_1$, follow “shifted-lognormal" stochastic-volatility processes

$$
\begin{aligned}
\mathrm{d}F_k^{x_1}(t) = \sigma_k^{x_1}(t) V^F(t) \Big[\frac{1}{\tau_k^{x_1}} + F_k^{x_1}(t) \Big] \\
\cdot \left[V^F(t) \sum_{h=\beta(t)}^{k} \rho_{h,k} \sigma_h^{x_1}(t) \,\mathrm{d}t + \mathrm{d}Z_k^{\mathcal{T}}(t) \right] \\
\mathrm{d}V^F(t) = a^F(t, V^F(t))\,\mathrm{d}t + b^F(t, V^F(t))\,\mathrm{d}W^{\mathcal{T}}(t)
\end{aligned}
$$

where:

- For each k, $\sigma_k^{x_1}$ is a deterministic function;
- $\{Z_1^{\mathcal{T}}, \ldots, Z_{M_1}^{\mathcal{T}}\}$ is an M_1-dimensional $Q_D^{\mathcal{T}}$-Brownian motion with correlations $(\rho_{k,j})_{k,j=1,\ldots,M_1}$;
- V^F is correlated with every $Z_k^{\mathcal{T}}$, $\mathrm{d}W^{\mathcal{T}}(t)\mathrm{d}Z_k^{\mathcal{T}}(t) = \rho_k^x \,\mathrm{d}t$, and $V^F(0) = 1$.

A tractable class of multi-tenor McLMMs

- The dynamics of forward rates F_k^x, for tenors $x \in \{x_2, \ldots, x_n\}$, can be obtained by Ito's lemma, noting that F_k^x can be written in terms of "smaller" rates $F_k^{x_1}$ as follows:

$$\prod_{h=i_{k-1}+1}^{i_k} [1 + \tau_h^{x_1} F_h^{x_1}(t)] = 1 + \tau_k^x F_k^x(t),$$

  for some indices i_{k-1} and i_k.
- We then assume, for each tenor $x \in \{x_1, \ldots, x_n\}$, the following one-factor models for the spreads:

$$S_k^x(t) = S_k^x(0)\mathcal{M}^x(t), \;\; k = 1, \ldots, M_x,$$

  where, for each x, $\mathcal{M}^x$ is a (continuous and) positive $Q_D^{\mathcal{T}}$-martingale independent of rates F_k^x and of the stochastic volatility V^F. Clearly, $\mathcal{M}^x(0) = 1$.

A tractable class of multi-tenor McLMMs

Rate dynamics under the associated forward measure

- When moving from measure $Q_D^{\mathcal{T}}$ to measure $Q_D^{T_k^x}$, the drift of a (continuous) process X changes according to

$$\textbf{Drift}(X; Q_D^{T_k^x}) = \textbf{Drift}(X; Q_D^{\mathcal{T}}) + \frac{\mathrm{d}\langle X, \ln(P_D(\cdot, T_k^x)/B_D^{\mathcal{T}}(\cdot))\rangle_t}{\mathrm{d}t}$$

- Applying Ito's lemma, we get, for each $x \in \{x_1, \ldots, x_n\}$,

$$\mathrm{d}F_k^x(t) = \sigma_k^x(t) V^F(t) \left[\frac{1}{\tau_k^x} + F_k^x(t) \right] \mathrm{d}Z_k^{k,x}(t)$$

where σ_k^x, $x \in \{x_2, \ldots, x_n\}$, is a deterministic function, whose value is determined by $\sigma_h^{x_1}$ and $\rho_{h,k}$, and

$$\mathrm{d}V^F(t) = -V^F(t) b^F(t, V^F(t)) \sum_{h=\beta(t)}^{i_k} \sigma_h^{x_1}(t) \rho_h^{x_1} \, \mathrm{d}t + a^F(t, V^F(t))\, \mathrm{d}t + b^F(t, V^F(t))\, \mathrm{d}W^{k,x}(t)$$

A tractable class of multi-tenor McLMMs

- The above dynamics of F_k^x are the simplest stochastic volatility dynamics that are consistent across different tenors x.
- If 3m-rates follow shifted-lognormal processes with common stochastic volatility, the same type of dynamics (modulo the drift correction in the volatility) is also followed by 6m-rates (under the respective forward measures).
- This allows us to price simultaneously, with the same type of formula, caps and swaptions with different tenors x.
- Option prices can then be calculated as suggested before. Swaption formulas can be simplified by noting that:

$$\bar{S}(t) = \sum_{k=a+1}^{b} \omega_k(t) S_k^x(0) \mathcal{M}^x(t)$$

$$\approx \mathcal{M}^x(t) \sum_{k=a+1}^{b} \omega_k(0) S_k^x(0) = \bar{S}(0) \mathcal{M}^x(t)$$

A tractable class of multi-tenor McLMMss

An explicit example of rate and spread dynamics

- We now assume constant volatilities $\sigma_k^{x_1}(t) = \sigma_k^{x_1}$ and SABR dynamics for V^F. This leads to the following dynamics for the x-tenor rate F_k^x under $Q_D^{T_k^x}$:

$$\mathrm{d}F_k^x(t) = \sigma_k^x V^F(t)\Big[\frac{1}{\tau_k^x} + F_k^x(t)\Big]\mathrm{d}Z_k^{k,x}(t)$$

$$\mathrm{d}V^F(t) = -\epsilon[V^F(t)]^2 \sum_{h=\beta(t)}^{i_k} \sigma_h^{x_1}\rho_h^{x_1}\mathrm{d}t + \epsilon V^F(t)\,\mathrm{d}W^{k,x}(t),$$

  with $V^F(0) = 1$, where also σ_k^x is now constant and $\epsilon \in \mathbb{R}^+$.
- We then assume that basis spreads for all tenors x are governed by the same geometric Brownian motion:

$$\mathcal{M}^x \equiv \mathcal{M}, \quad \mathrm{d}\mathcal{M}(t) = \sigma\mathcal{M}(t)\,\mathrm{d}Z(t)$$

  where Z is a $Q_D^{T_k^x}$-Brownian motion independent of $Z_k^{k,x}$ and $W^{k,x}$ and σ is a positive constant.

A tractable class of multi-tenor McLMMss

An explicit example of rate and spread dynamics

- Caplet prices can easily be calculated as soon as we smartly approximate the drift term of V^F. We get:

$$
\begin{aligned}
\mathbf{Cplt}(t, K; T_{k-1}^x, T_k^x) &= \int_{-\infty}^{a_k^x(t)} \mathbf{Cplt}^{\mathrm{SABR}}\Big(t, F_k^x(t) + \frac{1}{\tau_k^x}, K + \frac{1}{\tau_k^x} \\
&\quad - S_k^x(t) e^{-\frac{1}{2}\sigma^2 T_{k-1}^x + \sigma\sqrt{T_{k-1}^x} z}; T_{k-1}^x, T_k^x\Big) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\, \mathrm{d}z \\
&\quad + \tau_k^x P_D(t, T_k^x)(F_k^x(t) - K)\Phi\big(-a_k^x(t)\big) \\
&\quad + \tau_k^x P_D(t, T_k^x) S_k^x(t)\Phi\Big(-a_k^x(t) + \sigma\sqrt{T_{k-1}^x - t}\Big)
\end{aligned}
$$

where

$$
a_k^x(t) := \left(\ln \frac{K + \frac{1}{\tau_k^x}}{S_k^x(t)} + \frac{1}{2}\sigma^2(T_{k-1}^x - t)\right) \Big/ \left(\sigma\sqrt{T_{k-1}^x - t}\right)
$$

and the SABR parameters are σ_k^x (corrected for the drift approximation), ϵ and ρ_k^x (the SABR β is here equal to 1).

A tractable class of multi-tenor McLMMss

An explicit example of rate and spread dynamics

- This caplet pricing formula can be used to price caps on any tenor x.
- In fact, cap prices on a non-standard tenor z can be derived by calibrating the market prices of standard y-tenor caps using the formula with $x = y$ and assuming a specific correlation structure $\rho_{i,j}$.
- One then obtains in output the model parameters:
  - $\sigma_k^{x_1}$, $k = 1, \ldots, M_1$
  - $\rho_k^{x_1}$, $k = 1, \ldots, M_1$
  - ϵ
  - σ
- Finally, with these calibrated parameters one can price z-based caps, using again the caplet formula above, this time setting $x = z$.

A tractable class of multi-tenor McLMMss

An explicit example of rate and spread dynamics

- We finally consider an example of calibration to market data of this multi-tenor McLMM.
- We use EUR data as of September 15th, 2010 and calibrate 6-month caps with (semi-annual) maturities from 3 to 10 years. The considered strikes range from 2% to 7%.
- We minimize the sum of squared relative differences between model and market prices.
- We assume that OIS rates are perfectly correlated with one another, that all $\rho_k^{x_1}$ are equal to the same ρ and that the drift of V^F is approximately linear in V^F.
- The average of the absolute values of these differences is 19bp.
- After calibrating the model parameters to caps with $x = 6m$, we can apply the same model to price caps based on the 3m-LIBOR ($x = 3m$), where we assume that $\sigma_{i_{k-1}}^{3m} = \sigma_{i_k}^{3m}$ for each k.

A tractable class of multi-tenor McLMMss

An explicit example of rate and spread dynamics

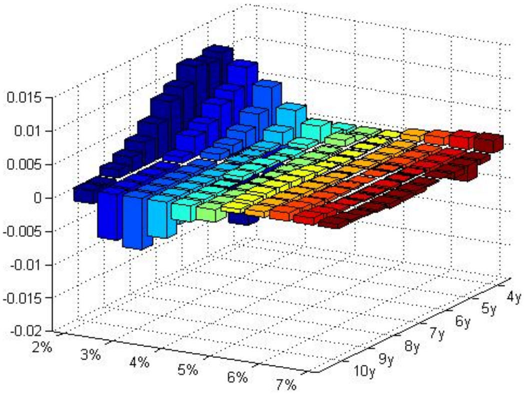


Figure: Absolute differences (in%) between market and model cap volatilities.

A tractable class of multi-tenor McLMMss

An explicit example of rate and spread dynamics

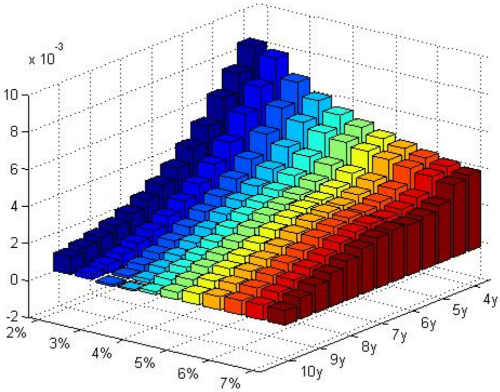


Figure: Absolute differences (in bp) between model-implied 3m-LIBOR cap volatilities and model 6m-LIBOR ones.

Conclusions

- We started by describing the changes in market interest rate quotes which have occurred since August 2007.
- We have shown how to price the main interest rate derivatives under the assumption of distinct curves for generating future LIBOR rates and for discounting.
- We have then shown how to extend the LMM to the multi-curve case, retaining the tractability of the classic single-curve LMM.
- We have finally introduced an extended LMM, where we jointly model rates and spreads with different tenors.
- References:

  - Mercurio, F. (2010a) Modern LIBOR Market Models: Using Different Curves for Projecting Rates and for Discounting. *International Journal of Theoretical and Applied Finance* 13, 1-25.
  - Mercurio, F. (2010b) LIBOR Market Models with Stochastic Basis. Available online on the ssrn web site.