Capital Conservation and Risk Management

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Basic background [Cherny and Madan '09, '10]

- **Needed:** Framework to study capital conservation, risk management and hedging in illiquid derivative markets.

  $\Rrightarrow$ Illiquid derivative markets as competitive counterparties creating new financial products and efficiently using liquid hedging instruments.

  $\Rrightarrow$ Ask and bid prices reflect the cost of holding unhedgeable risk, rather than processing, inventory or transaction costs.

- **Approach:** Convex cone $\mathcal{A}$ of acceptable cash-flows:

$$X \in \mathcal{A} \Leftrightarrow E^{Q}(X) \geq 0 \text{ for all } Q \in \mathcal{M} \tag{1}$$

  for some convex set $\mathcal{M}$ of measures equivalent to P [Artzner et. all '99].

- **Liquid hedging instruments**: Modeled as a vector space $\mathcal{H}$, given a set $\mathcal{R}$ of risk-neutral measures equivalent to P:

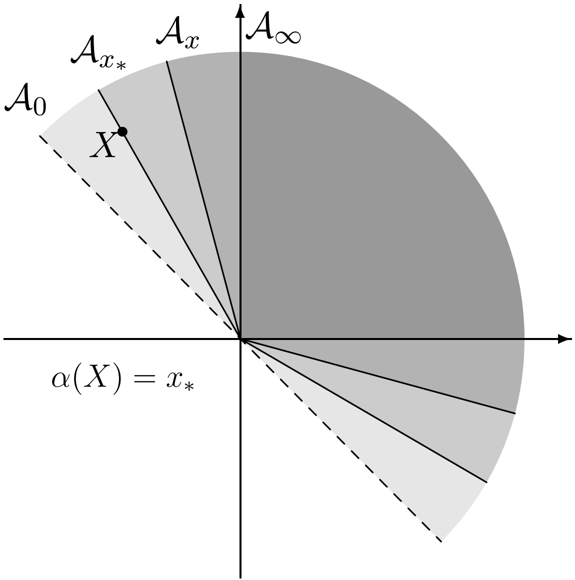
$$H \in \mathcal{H} \Leftrightarrow E^{Q}(H) = 0 \text{ for all } Q \in \mathcal{R}\ . \tag{2}$$

- **Competitive bid-ask spread**: Modeled through $\mathcal{M}$ and $\mathcal{R}$:

$$\begin{aligned} a(X) &= \inf\{a : a + H - X \in \mathcal{A} \text{ for some } H \in \mathcal{H}\} = \sup_{Q \in \mathcal{M} \cap \mathcal{R}} E^{Q}(X) \\ b(X) &= \sup\{b : -b - H + X \in \mathcal{A} \text{ for some } H \in \mathcal{H}\} = \inf_{Q \in \mathcal{M} \cap \mathcal{R}} E^{Q}(X) \end{aligned}$$

  Distinct, e.g., from superhedging-type approaches.

Convex cone $\mathcal{A}$ of market-acceptable cash flows



Concave distortions [Cherny and Madan '09, '10]

- **Model of market acceptable cash flows**: Given distribution function $F_X(x)$,

$$X \in \mathcal{A} \Leftrightarrow E^Q(X) \geq 0 \text{ for all } Q \in \mathcal{M} \Leftrightarrow \int x d(\Psi \circ F_X)(x) \geq 0$$

  where $\Psi(u)$ is a concave distribution on $[0, 1]$.

  $\Rrightarrow$ Convex set $\mathcal{M}$ is fully characterized in terms of Ψ [Cherny '06].

- **Density** $\psi(x) := (\Psi' \circ F)(x)$ **with respect to original measure** P:

  $\Rrightarrow$ $\Psi' \circ F_X$ defines market-preferences by a "stressed" distribution that shifts probability mass towards negative cash flows.

  $\Rrightarrow$ Like utility kernels, $\Psi' \circ F_X$ can be taken to put arbitrarily large (small) mass on large negative (positive) cash flows [e.g., for MINMAXVAR Ψ's]

- **Parametric bid and ask**:

$$\begin{aligned} a(X) &= \inf\{a : a + \int x d(\Psi \circ F_{H-X})(x) \geq 0 \text{ for some } H \in \mathcal{H}\} \\ &= \inf_{H \in \mathcal{H}} -\int x d(\Psi \circ F_{H-X})(x) \qquad (3) \\ b(X) &= \sup\{b : -b + \int x d(\Psi \circ F_{X-H})(x) \geq 0 \text{ for some } H \in \mathcal{H}\} \\ &= \sup_{H \in \mathcal{H}} \int x d(\Psi \circ F_{X-H})(x) \qquad (4) \end{aligned}$$

Example: Stressed densities $\Psi' \circ F_X$

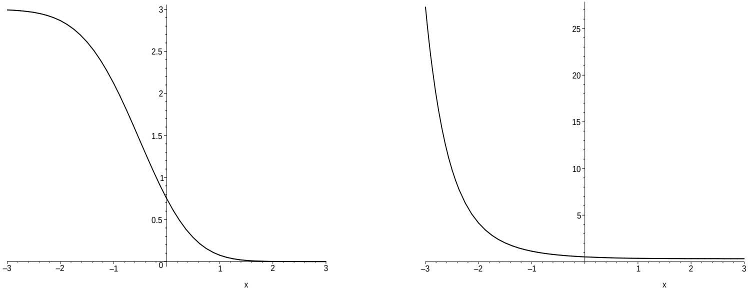


Figure 2. **(a)** Extreme measure densities for $\Psi(x) = 1-(1-x)^3$.

(b) Extreme measure densities for $\Psi(x) = x^{1/3}$.

- **MINVAR** [$\Psi_\gamma(u) = 1-(1-u)^{1+\gamma}$]: implies an infinity (zero) mass at large negative (positive) cash flows values.
- **MAXVAR** [$\Psi_\gamma(u) = u^{1/(1+\gamma)}$]: implies a bounded (zero) mass at large negative (positive) cash flows values.
- **MINMAXVAR** [$\Psi_\gamma(u) = 1-(1-u^{1/(1+\gamma)})^{1+\gamma}$]: implies an infinity (zero) mass at large negative (positive) cash flows values.

Quantile exposures and risk charges [Carr et al. '10]

- **Idea:** Split the price of a contingent payoff into (i) a quantile exposure and (ii) a charge for quantile risk.
- **Bid and ask prices:** Given in terms of the inverse distribution function $G_H(u)$ of a hedged cash flow $X - H$ with median $m = G_H(1/2)$:

$$a(X) = m + \inf_{H \in \mathcal{H}} \int_0^1 \left[\Psi(1-u) - \mathbb{I}(u \leq 1/2)\right] dG_H(u)$$

$$b(X) = m + \sup_{H \in \mathcal{H}} \int_0^1 \left[\mathbb{I}(u \geq 1/2) - \Psi(u)\right] dG_H(u)$$

- $dG_H(u)$ is the sensitivity of the cash flow to a change in the quantile:
  $\Rrightarrow$ It gives the risk exposure of that particular quantile under distribution $F_H(x)$.
- Over interval $dG_H(u)$, the charge for ask and bid prices is:

$$\Psi(1-u) - \mathbb{I}(u \leq 1/2) \quad ; \quad \mathbb{I}(u \geq 1/2) - \Psi(u) \tag{5}$$

  $\Rrightarrow$ Equation (5) defines the $\Psi-$dependent risk charge per unit of quantile risk exposure.
- Similar interpretations for bid-ask related quantities, like capital, profit, etc., see below.

Profit, capital and leverage [Carr et al. '10]

- **Capital**: Cost of unwinding a position, i.e., the bis-ask spread:

$$k(X) = a(X) - b(X) = \int_0^1 K(u)dG(u)$$

  where $K(u)$ is symmetric about 1/2.

- **Profit** [given fixed risk neutral probability P]:
  - Market distributes half of bid-ask spread to market participants.
  - Cash flow production cost is its risk neutral expectation.

$$\begin{aligned} \pi(X) &:= m(X) - c(X) \\ &:= \frac{a(X) + b(X)}{2} - E^P(X) = \int_0^1 H(u)dG(u) \end{aligned}$$

  where $H(u)$ is antisymmetric about 1/2.

- **Rate of return**:

$$\rho(X) := \pi(X)/k(X)$$

- **Scale**: Translation-invariant measure of scale of operations (associated with leverage to be granted for given capital $k(X)$):

$$scale(X) := E^P(|X - m(X)|) = \int_0^1 S(u)dG(u)$$

Profit and capital charges [$H(u)$, $K(u)$]

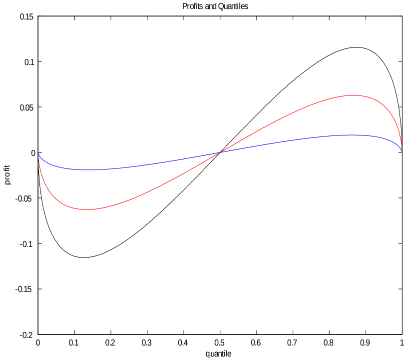


Figure 1: The profit charge on quantiles for MINMAXVAR at three stress levels of 0.1, 0.25 and 0.5

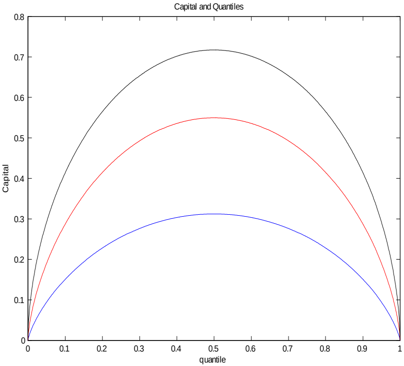


Figure 2: Capital charges for different quantile levels for MINMAXVAR at three stress levels of 0.1, 0.25 and 0.5.

Capital vs. scale charges

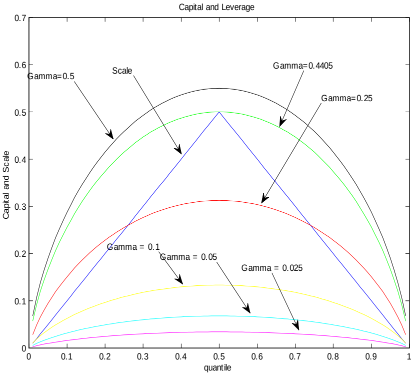


Figure 3: Graph of Capital Charges against Scale for various settings of the stress parameter in minmaxvar.

Applications

- **Variance-swap hedging**: [Illiquid markets with (skewed) VG underlying]

  ⇛ Standard hedge reduces bid-ask spreads and raises returns on earlier maturities.

  ⇛ Standard hedge produces losses on longer maturities, due to a larger unhedged cash flow risk.

  ⇛ A hedge minimizing first the ask and then the capital committed can avoid the lossed of the standard hedge.

- **Call option hedging**: [left skewed VG underlying]

  ⇛ Capital minimization is not well achieved by expected utility optimization.

- **Delta hedging**: [left skewed (VG) returns]

  ⇛ Under concave distortion $\Psi(u)$ downside risk is more heavily priced than upside risk.

  ⇛ To minimize capital, the optimal delta should be revised downwards in presence of Γ exposure.

- **Dynamic extensions via dynamically consistent non-linear expectations** [Thm 6.1, Cohen and Elliott, '10]:

  ⇛ Solution of backward stochastic difference equation with corresponding driver:

$$Y_t^j = E_t[Y_{t+1}^j] + \int_{-\infty}^{\infty} x d(\Psi \circ \Theta_t^j)(x) \tag{6}$$

where Θ_t^j is the distribution function of $Y_{t+1}^j - E_t[Y_{t+1}^j]$, $j = bid, ask$.

Comments (I)

Model of financial market as competitive capital optimizer: Aspects...

- **General:**
  - Largely based on univariate hedging problems (because of law invariance), thus abstracting from potential portfolio dependencies (centralized vs. decentralized markets; exchanges vs. over-the-counter)?
  - Can the approach be reconciled with demand pressure effects documented in, e.g., index and individual option markets [Garleanu et al. '09]?
  - Concrete specifications implicitly linked to parametric assumptions on "market-preferences" via chosen distortion $\Psi(u)$ (i.e., cone $\mathcal{A}$).
    $\Rrightarrow$ How to identify $F(x)$ and $\Psi(u)$ only from cross-sectional information without parametric assumptions?
    $\Rrightarrow$ Not always clear in the draft whether this is with respect to risk-neutral or physical probabilities...
    $\Rrightarrow$ Time-series information might help to separate probabilistic cash flow features from market-driven price distortions?
  - Definition of profits related to cash flow "replication costs" in incomplete markets; uniquely defined?
  - Deeper interpretation of (virtual) assumption that profits are evenly redistributed in competitive markets? How could this effectively function?

Comments (II)

Model of financial market as competitive capital optimizer: Aspects...

- **Some (among many) potential applications:**
  - Joint explanations of bid and ask prices of, e.g., put and call option smiles? Comparison to fit of standard approaches?
  - Time variation of bid ask spreads in terms of time variation in implied distortions:
    ⇛ Joint cross-sectional and time series study!?
    ⇛ Proxies of time-varying market fear, e.g., linked to time-varying uncertainty or uncertainty aversion!?
    ⇛ Deeper implied (possibly multivariate) liquidity-market depth proxies in terms of estimated cone of acceptable cash flows?
- **Overall, very interesting framework to study a variety of questions in illiquid financial markets**!

Appendix I: MINMAXVAR features [Cherny '06]

- **MINMAXVAR as weighted Tail VAR (WVAR)**:

$$WVAR_{\mu}(X) = \int_{(0,1]} TVAR_{\lambda}\mu(d\lambda) \tag{7}$$

given measure μ on $(0,1]$ and tail Value at Risk $TVAR_{\lambda} = -E[X|X \leq q_{\lambda}(X)]$.

- **Föllmer and Schied '04:** One-to-one relation between concave distortions and measures on $(0,1]$:

$$\begin{aligned} WVAR_{\mu}(X) &= -\int_{(0,1]} \left(\lambda^{-1} \int_{(-\infty, q_{\lambda}(X)]} y dF_X(y) \right) \mu(d\lambda) \\ &= -\int_{\mathbb{R}} y \left(\int_{(F_X(y),1]} \lambda^{-1}\mu(d\lambda) \right) dF_X(y) \\ &= -\int_{\mathbb{R}} y d(\Psi_{\mu} \circ F_X)(y) \end{aligned} \tag{8}$$

where $\Psi_{\mu}(u) := \int_0^u \int_{(z,1]} \lambda^{-1}\mu(d\lambda)dz$.