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Dynamical Counterparty Risk Valuation via Bessel Bridges

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AGENDA

- Combining pricing and counterparty default models
- Calibrating firm-value default models to CDS data
- Solving inverse hitting time distribution problems
- Conditional Brownian motion and Bessel bridges
- Applications to vanilla options and interest-rate swaps

Counterparty Risk

This is the exposure of a bank to a counterparty in some contract should the counterparty default *at some specific time* in the future.

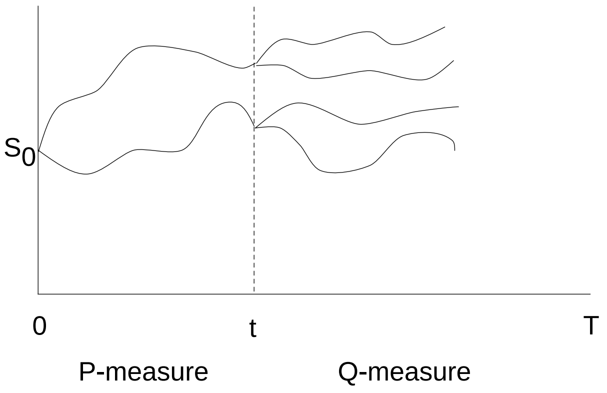
Note: Exposure = max(value, 0) [No exposure if we owe them!]

'Exposure' can be quantified in various ways:

- Quantile of loss distribution
- Expected shortfall
- $\cdots$

These are all functions of the post-default distribution of contract value.

Example: Long contract in (Black-Scholes) call option.
  Value now: $\mathrm{BS}(S_0, K, r, \sigma, T)$
  Value at $t \in (0, T)$: $V_t = \mathrm{BS}(S_t, K, r, \sigma, T - t)$
so problem amounts to computing the distribution of S_t.



Note this is in principle a 'two-measure problem':

- Distribution of S_t in 'real-world' measure $\mathbb{P}$
- $V_t = \mathbb{E}_{\mathbb{Q}}[e^{-r(T-t)}(S_t - K)^+ | \mathcal{F}_t]$ where $\mathbb{Q}$ is risk-neutral measure.

Nevertheless, computations are normally done using only $\mathbb{Q}$-measure, because

- It is absolutely impossible to predict the $\mathbb{P}$-measure drift.
- For short horizons t, volatility, not drift, is the dominant factor.

Right-way/Wrong-way risk

The simplest approach is just to compute the distribution of S_t (under $\mathbb{Q}$, say). In the Black-Scholes model we have

$$S_t = S_0 \exp((r - \frac{1}{2}\sigma^2)t + \sigma W_t)$$

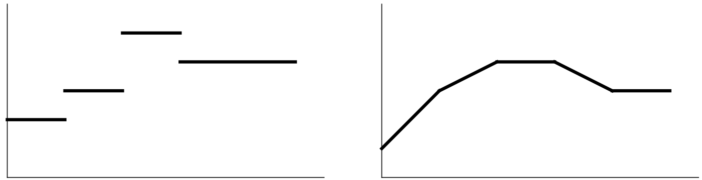
where W is BM, i.e. $W_t \sim N(0,t)$. However, this ignores any connection between exposure and the assumed counterparty default at t. We want the *conditional distribution* given that the default event occurs at t.

Right-way risk: Negative correlation between default and exposure.

Wrong-way risk: Positive correlation.

We first need a counterparty default model. We take the approach of John Hull: default time τ is the first hitting time of some barrier by BM.

Barrier calibration (John Hull)

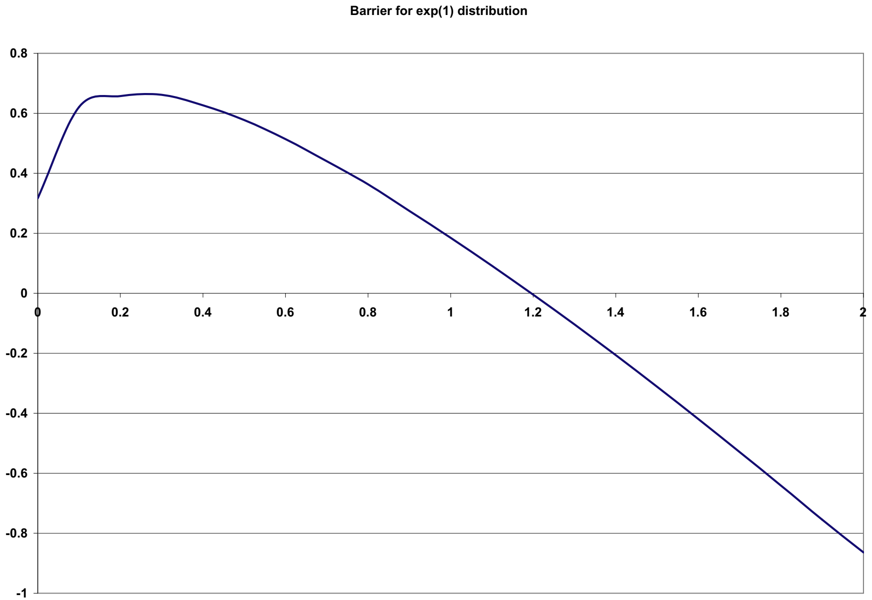


CDS (credit default swap) spreads determine the $\mathbb{Q}$-measure distribution function F of the counterparty default time τ.

Calibrate a piecewise-constant barrier (levels $b_0, b_1, \ldots$) or a piecewise-linear barrier (level b_0 and slopes $d_0, d_1, \ldots$) so that F is perfectly matched at discrete time points. Procedure for pw-constant barrier: choose b_0 so that hitting prob on $[0, t_1]$ is $F(t_1)$. Now get a discrete representation of the distribution of B_{t_1} on $(-\infty, b_0)$ given that the barrier is not hit, and bootstrap for $b_1, b_2, \ldots$.

Practical point: Much better to use $X_t = B_t + \nu t$ with $\nu > 0$ (rotates barrier).

Barrier for exponential hitting time



Application to counterparty risk in Black-Scholes

As above, price process is

$$S_t = S_0 \exp((r - \frac{1}{2}\sigma^2)t + \sigma W_t)$$

Assume default is controlled by BM B_t as above, and

$$W_t = \rho B_t + \sqrt{1 - \rho^2} B'_t$$

where B, B' are independent. Suppose default takes place at $t = 0.25$. Then $B_t = b(t)$ so $W_t = \rho b(t) + \sqrt{1 - \rho^2} B'_t$, i.e.

$$W_t \sim N(\rho b(t), (1 - \rho^2)t).$$

Graph shows corresponding distribution of option value. (In this case $b(t) = 0.5$.)

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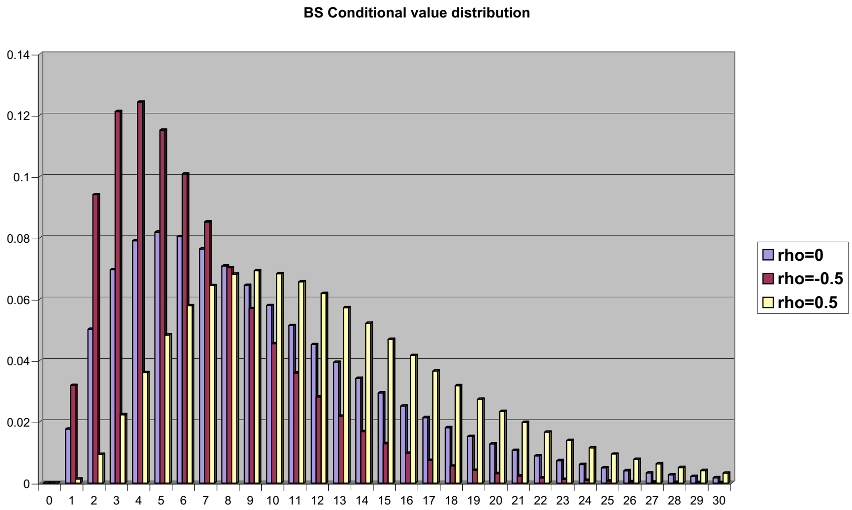
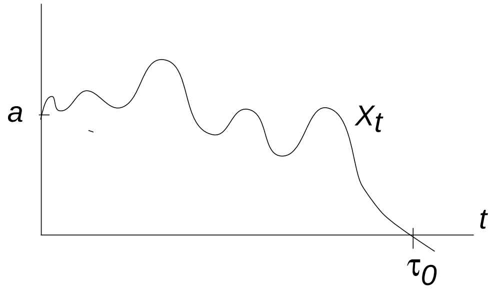


BS value distribution with CP default at $t = 0.25$

$\rho = 0$ (blue), $\rho = .5$ (yellow), $\rho = -.5$ (red).

An alternative approach

The idea: instead of calibrating with a time-varying barrier, keep a *flat* barrier and vary the Brownian motion in some way. Always start at $a > 0$ and take 0 as the barrier.



Define $\tau_0^X = \inf\{t : X_t \leq 0\}$ with Brownian motion B_t and

$$X_t = a + \nu t + B_t.$$

Then τ_0^X has distribution function K with density

$$k(t) = \frac{d}{dt}K(t) = \frac{a}{\sqrt{2\pi t^3}} \exp\left(-\frac{(a+\nu t)^2}{2t}\right).$$

$$K(\infty) = \mathbb{P}[\tau_0^X < \infty] = \begin{cases} 1, & \nu \le 0 \\ e^{-2\nu a}, & \nu > 0 \end{cases} .$$

The *moment generating function* is given by

$$\mathbb{E}_a[e^{-q\tau_0^X}] = e^{-a\,\Phi(q)}, \quad q > 0, \tag{1}$$

where $\Phi(q) = \nu + \sqrt{\nu^2 + 2q}$. This is shown as follows. For $\theta \in \mathbb{R}$, define

$$\begin{aligned} Z_t &= \theta B_t - \frac{1}{2}\theta^2 t \\ &= \theta(X_t - a) - (\nu\theta + \frac{1}{2}\theta^2)t. \end{aligned}$$

For $q > 0$ the equation $\theta^2/2 + \theta\nu = q$ has negative root $\theta = -\Phi(q)$ with Φ as above. Then $\exp(Z_{t\wedge\tau_0^X})$ is a bounded martingale and we conclude that

$$1 = \mathbb{E}_a[\exp(Z_{\tau_0^X})] = e^{a\,\Phi(q)}\mathbb{E}_a[e^{-q\,\tau_0^X}].$$

1. Fixed starting point

Recall

$$X_t = a + \nu t + B_t. \tag{2}$$

Let $\sigma(t)$ be a (deterministic) non-negative function and define

$$I_t = \int_0^t \sigma^2(s)ds.$$

Then $X(I_t)$ is equal in law to the process Y_t defined for some BM $\tilde{B}_t$ by

$$dY_t = \nu\sigma^2(t)dt + \sigma(t)d\tilde{B}_t, \qquad Y_0 = a. \tag{3}$$

$\tau_0^Y \equiv \inf\{t : Y_t = 0\}$ is our model for the default time.

Theorem 1. Let H be a distribution function on $\mathbb{R}^+ \cup \{+\infty\}$ with $H(0) = 0$, having a density function h. In (2), fix $a > 0$ and

$$\nu < -\frac{1}{2a} \log H(\infty), \tag{4}$$

and define Y_t by (11) with

$$\sigma^2(s) = \begin{cases} 0, & 0 \le s \le \inf\{u : H(u) > 0\} \\ \frac{h(s)}{k(K^{-1}(H(s)))}, & s > s_0. \end{cases} \tag{5}$$

Then $\mathbb{P}[\tau^Y \le t] = H(t),\ t \in \mathbb{R}^+$.

Proof: With $\sigma^2(s)$ defined by (5) we find that

$$I_t = K^{-1}(H(t)).$$

This is well-defined because condition (4) ensures that $K(\infty) \ge H(\infty)$. Thus

$$\mathbb{P}[\tau_0^Y \le t] = \mathbb{P}[\tau_0^X \le I_t] = K(I_t) = H(t).$$

Conclusion: we can realize any distribution on $\mathbb{R}^+$ having a density as the hitting time of zero by a drifting BM with deterministic time change.

2. Randomized starting point

We now introduce an extra degree of flexibility by taking $X_0 = A$, where A is a random variable having distribution function F, independent of $B(\cdot)$.

We ask: given a distribution function H on $\mathbb{R}^+ \cup \{\infty\}$ with density function h, can we choose F on such that τ_0^X has distribution H when

$$X_t = A + \nu t + B_t.$$

Such an F must satisfy

$$H(t) = \mathbb{P}[\tau_0 \leq t] = \int_0^\infty \mathbb{P}_a[\tau_0^X \leq t] F(da).$$

Taking the Laplace-Stieltjes transform in t, $\mathcal{L}H(q) = \int_{\mathbb{R}^+} e^{-qt} H(dt)$, gives

$$\mathcal{L}H(q) = \int_0^\infty \mathbb{E}_a[e^{-q\tau_0^X}] F(da) = \int_0^\infty e^{-a\,\Phi(q)} F(da).$$

We know Φ has inverse $q = \psi(\theta) = \frac{1}{2}\theta^2 - \nu\theta$ so if F exists it must satisfy

$$\mathcal{L}H(\psi(\theta)) = \mathcal{L}F(\theta) \tag{6}$$

Important example: Exponential distribution $H(t) = 1 - e^{-\lambda t}$

Here the left-hand side of (6) is

$$\frac{\lambda}{\psi(\theta)+\lambda} = \frac{2\lambda}{\theta_+ - \theta_-}\left(\frac{1}{\theta-\theta_+} - \frac{1}{\theta-\theta_-}\right).$$

This is the LT of a distribution on $\mathbb{R}^+$ if and only if $\psi(\theta) > \lambda$ for all $\theta \geq 0$ and $\nu^2 - \lambda \geq 0$. We obtain the following solutions for the density $f_\lambda(x) = (d/dx)F(x)$.

- $\nu < -\sqrt{2\lambda}$: $\quad f_\lambda(x) = \frac{2\lambda}{\theta_+-\theta_-}(e^{\theta_+ x} - e^{\theta_- x})$.
- $\nu = -\sqrt{2\lambda}$: $\quad f_\lambda(x) = 2\lambda x e^{-x\sqrt{2\lambda}}$.

Can we ‘realize’ an arbitrary density function h in this way? Answer: no. If a solution exists then the Laplace transform of the corresponding F must be given by the LHS of (6), but this formula may not correspond to a distribution having support in $\mathbb{R}^+$.

To go further, introduce a deterministic time change I_t as before, where

$$I_t = \int_0^t \sigma^2(s)ds$$

and $X(I_t) =_d Y_t$ with

$$dY_t = \nu\sigma^2(t)dt + \sigma(t)d\tilde{B}_t, \quad Y_0 = A. \tag{7}$$

Essentially we realize an exponential distribution as above and then massage the time scale to get the distribution we want. Thus we choose $A \sim F$ where $(dF/dx)(x) = f_\lambda(x)$ as above, so that when $\sigma \equiv 1$ then $\tau_0^Y \sim \exp(\lambda)$.

Theorem 2. Let H be a distribution on $\mathbb{R}^+$ with density h and hazard function

$$\gamma(t) = \frac{h(t)}{\int_t^\infty h(s)ds} = \frac{h(t)}{\overline{H}(t)}.$$

Define Y_t by (7) with

$$\sigma(t) = \sqrt{\frac{\gamma(t)}{\lambda}}.$$

Then $\tau_0^Y \sim H$ where

$$\tau_0^Y = \inf\{t : Y_t = 0\}.$$

Indeed, since $h/\overline{H} = -(d/dt)\log \overline{H}$ we have

$$I_t = -\frac{1}{\lambda}\log \overline{H}(t).$$

Since $Y_t = X(I_t)$ and τ_0^X has exponential distribution, we see that

$$\mathbb{P}[\tau_0^Y > t] = \mathbb{P}[\tau_0^X > I_t] = e^{-\lambda I_t} = e^{\log \overline{H}(t)} = \overline{H}(t).$$

Calibrating Risk-neutral default-time distributions from CDS Rates

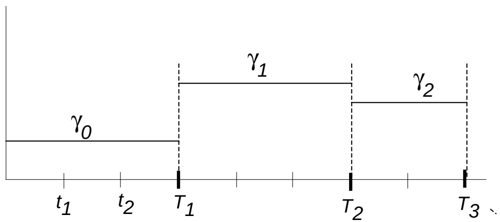
For CDS contracts written on an underlying name ABC, we assume that premium payments are made at times t_i and the available maturities are $T_j = t_{k(j)}, j = 1, \ldots, n$. For contract j there is an upfront premium π_j^0 and a running premium rate π_j^1 (with accrual factors δ_i). The recovery rate is $R \in (0,1)$. The 'fair premium' (π_j^0, π_j^1) then satisfies

$$\pi_j^0 + \pi_j^1 \sum_{i=0}^{k(j)-1} \delta_i p(0,t_i)\overline{H}(t_i) = (1-R) \sum_{i=1}^{k(j)} p(0,t_i)(\overline{H}(t_{i-1}) - \overline{H}(t_i)). \quad (8)$$

We take the default distribution to have piecewise-constant hazard rate, i.e.

$$\overline{H}(t) = \exp\left(- \int_0^t \gamma(s) ds \right)$$

where $\gamma(s) = \gamma_i$ for $T_i \le s < T_{i+1}$ (with $T_0 = 0$.)



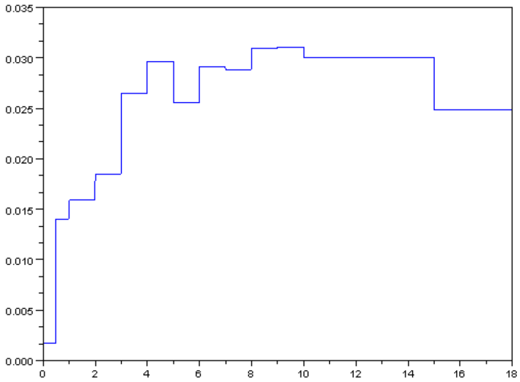
We then back out $\gamma_0, \gamma_1, \ldots$ from (8) given the market data $(\pi_1^0, \pi_1^1), (\pi_2^0, \pi_2^1), \ldots$

Note this fits perfectly with our default model, which is hitting of 0 by Y_t satisfying

$$dY_t = \nu \sigma^2(t) dt + \sigma(t) d\tilde{B}_t, \quad Y_0 = A,$$

where $\sigma^2(t) = \gamma(t)/\lambda$, i.e. Y_t has piecewise-constant coefficients.

Example: Altria Group Inc



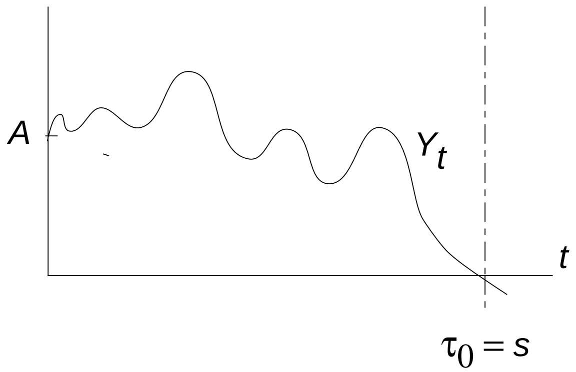
We have 12 CDS quote for maturities ranging from 6m to 16y.

Models conditioned on default time

To evaluate counterparty risk, we condition on default at a specific time $s > 0$. For path-dependent contracts we need the conditional law of the default risk process Y_t conditioned on the event $(\tau_0^Y = s)$. We consider

1. Brownian motion case.

2. Case of general calibrated model as above.



Bessel Bridges, h-transforms etc.

The law of Brownian motion B starting at $a > 0$ and conditioned to hit 0 for the first time at $\tau_0 = s$ is equal to that of the 3-dimensional Bessel Bridge from $a \to 0$ on $[0, s]$. We apply the Doob h-transform with h given by

$$\begin{aligned} h(t,x) &= \text{`}\mathbb{P}[\tau_0 = s | B_t = x]\text{'} \\ &= \mathbb{P}_x[\tau_0 \in [s - dt, s]]/dt \\ &= \frac{1}{\sqrt{2\pi(s-t)^3)}} x e^{-x^2/2(s-t)}. \end{aligned}$$

$h(t, B_t)$ is a positive martingale, so (with $h' = \partial h/\partial x$) $dh = h' dB = h(h'/h) dB$ and

$$\frac{h(t,B_t)}{h(0,a)} = \mathcal{E}\left(\frac{h'}{h} \cdot B\right).$$

We apply a change of measure $d\mathbb{Q}/d\mathbb{P}|_{\mathcal{F}_t} = h(t, B_t)/h(0, a)$. By Girsanov, the new drift is

$$\frac{h'}{h} = (\log h)' = \left(\log x - \frac{1}{2(s-t)} x^2\right)' = \frac{1}{x} - \frac{x}{s-t}.$$

Thus under $\mathbb{Q}$, X_t satisfies the SDE

$$dX_t = \left(\frac{1}{X_t} - \frac{X_t}{1-t} \right) dt + dB, \quad t \in [0, s) \tag{9}$$

Recall that the Brownian bridge Z_t from $Z_0 = a$ to $Z_1 = 0$ satisfies

$$dZ_t = -\frac{Z_t}{1-t} dt + dW_t, \quad Z_0 = a.$$

It can also be represented as

$$Z_t = \frac{s-t}{s} a + B_t - \frac{t}{s} B_s$$

where B_t is ordinary Brownian motion.

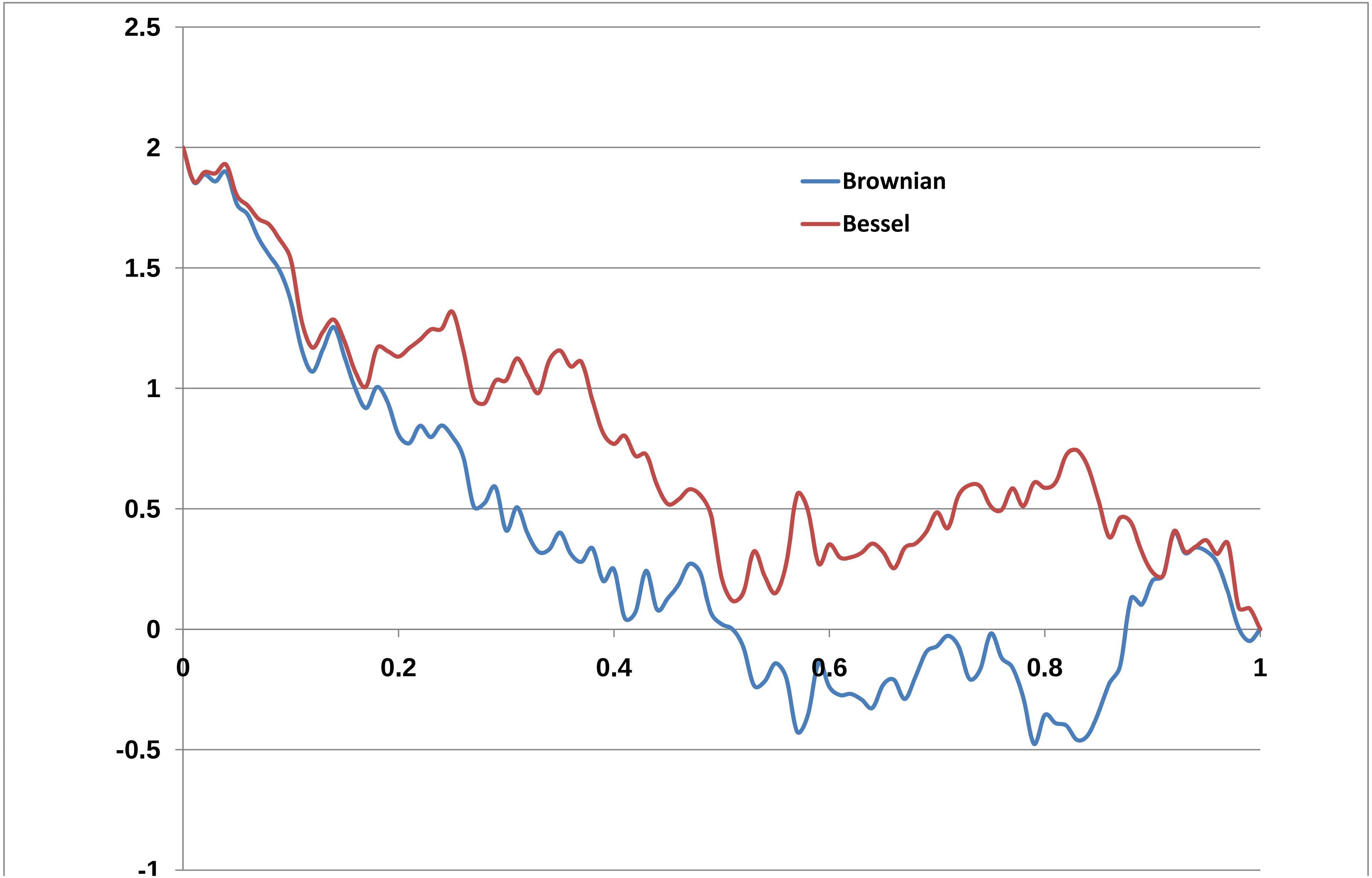
Taking a 3-vector Z_t of independent Brownian bridges and applying the Ito formula, we find that $|Z|$ satisfies (9), so $X =_{\mathcal{L}} |Z|$.

Finally, a result of Bertoin and Pitman states that

$$X = |_{\mathcal{L}} \quad \sqrt{(a(s-t)/s + X_{1,t})^2 + X_{2,t}^2 + X_{3,t}^2}$$

where X_i, $i = 1, 2, 3$ are independent $0 \to 0$ Brownian Bridges.

This provides us with an efficient simulation method.



2.5
2
1.5
1
0.5
0
-0.5
-1
0
0.2
0.4
0.6
0.8
1
Brownian
Bessel

General case

Our main result is as follows. Recall that our general default time model is τ_0^Y, the first hitting time of 0 by the process

$$Y_t = \nu\sigma^2(t) + \sigma(t)dB_t, \quad Y_0 = A.$$

Proposition. Conditioned on $\tau_0^Y = s > 0$, the process Y_t satisfies

$$\begin{aligned} dY_t &= \left(\frac{1}{Y_t} - \frac{Y_t}{\int_t^s \sigma^2(u)du}\right)\sigma^2(t)dt + \sigma(t)d\tilde{B}_t, \quad t \in (0,s) \\ Y_0 &= A, \end{aligned}$$

where $\tilde{B}$ is Brownian motion and $A \sim F$ is independent of $\tilde{B}$.

The $(\cdots)$ term can alternatively be expressed as

$$\left(\frac{1}{Y_t} - \frac{\lambda Y_t}{\log(\overline{H}(t)/\overline{H}(s))}\right).$$

Interest rate swaps

The zero-coupon (ZC) bond $p(t, T)$ gives the value at $t \leq T$ of £1 delivered at T (so $p(T, T) = 1$). A *short rate model* specifies the ZC bond price as

$$p(t,T) = \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r(s)ds} \middle| \mathcal{F}_t \right],$$

where $r(t)$ is the short-rate process. Here we consider the Hull-White model

$$dr(t) = (\theta(t) - \mu r(t))dt + \beta dW_t.$$

This is an 'affine process' in that the ZC bond value takes the form

$$p(t,T) = \exp(A(t,T) - B(t,T)r(t)) \equiv p_{tT}(r(t)).$$

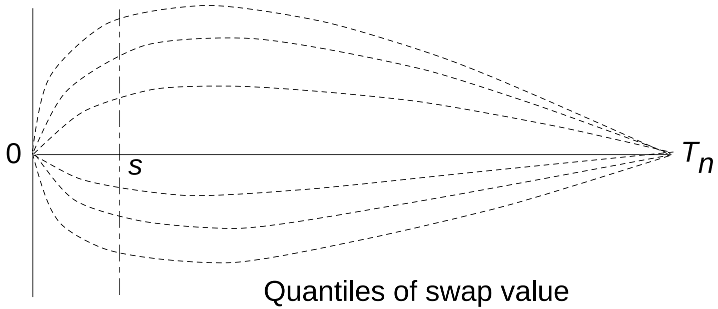
In an *interest rate swap* the parties exchange a fixed-rate payment $\delta_i K$ for a floating (Libor) payment $\delta_i L_i$ at times $T_1, , \ldots, T_n$ where δ_i is the accrual factor and L_i is the Libor rate set at T_{i-1}. The (payer's) swap value at any time is

$$V_t = p(t, T_{k(t)}) - p(t, T_n) - K \sum_{i \geq k(t)} \delta_i p(t, T_i)$$

where $k(t)$ is the next coupon date after t.

The *swap rate* S_t is the value of K such that $V_t = 0$, i.e.

$$S_t = \frac{p(t, T_{k(t)}) - p(t, T_n)}{\sum_{i \geq k(t)} \delta_i p(t, T_i)}.$$



Quantiles of swap value

Counterparty risk problem: Calculate swap value distribution at $t > 0$ conditional on counterparty default at t. Note V_t is a function of $r(t)$, a *path-dependent functional* so we can't use the simple Black-Scholes method.

Counterparty risk

The essential problem is to get the distribution of $r(s)$ given that default happens (i.e. flat barrier with random starting point) is hit at s. Recall

$$dr(t) = (\theta(t) - \mu r(t))dt + \beta dW_t. \tag{10}$$

We take the counterparty risk model developed above, i.e. the default time is τ_0^Y where Y is the process

$$dY_t = \nu\sigma^2(t)dt + \sigma(t)dB_t, \quad Y_0 = A. \tag{11}$$

Here W, B are Brownian motions with correlation ρ, i.e. $\mathbb{E}[dW\, dB] = \rho dt$, so we can represent W as

$$W_t = \rho B_t + \bar{\rho}\tilde{B}_t,$$

where $B, \tilde{B}$ are independent BMs and $\bar{\rho} = \sqrt{1-\rho^2}$.

Calibration: $\theta(\cdot)$ is calibrated from the swap market, (μ, β) from the swaption vol matrix and $\sigma(\cdot)$ from the counterparty CDS quotes.

The solution of (10) is

$$r(s) = \alpha(s) + \int_0^s e^{-\mu(s-u)} \rho\beta dB_s + \int_0^s e^{-\lambda(s-u)} \bar{\rho}\beta d\tilde{B}_s \tag{12}$$

where $\alpha(s)$ is the deterministic function

$$\alpha(s) = e^{-\mu s} r(0) + \int_0^s e^{-\mu(s-u)} \theta(u) du.$$

The third term in (12) is $\Xi(s) \sim N(0, \Sigma^2(s))$ where

$$\Sigma^2(s) = \frac{\bar{\rho}^2 \sigma^2}{2\lambda}(1 - e^{-2\lambda s}).$$

From (11) we have

$$dB = \frac{1}{\sigma(t)} dY - \nu\sigma(t) dt,$$

so the second term in (10) is

$$\int_0^s e^{-\mu(s-u)} \frac{\rho\beta}{\sigma(u)} dY(u) - \int_0^s e^{-\mu(s-u)} \rho\beta\nu\sigma(u) du.$$

In summary we have

$$\boxed{r(s) = \tilde{\alpha}(s) + \Xi(s) + \int_0^s e^{-\mu(s-u)} \frac{\rho\beta}{\sigma(u)} dY(u),} \tag{13}$$

where

$$\tilde{\alpha}(s) = \alpha(s) - \int_0^s e^{-\mu(s-u)} \rho\beta\nu\sigma(u)du.$$

If we *condition on default at time* s then Y_t satisfies the SDE

$$\boxed{dY_t = \left(\frac{1}{Y_t} - \frac{Y_t}{\int_t^s \sigma^2(u)du} \right) \sigma^2(t)dt + \sigma(t)d\tilde{B}_t, \quad t \in (0, s)} \tag{14}$$

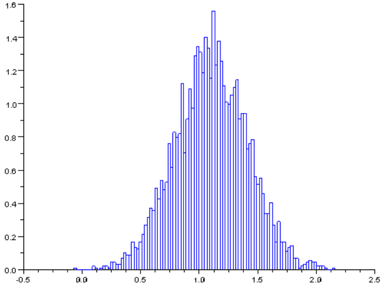
By Monte Carlo simulation of (13),(14) we can obtain the empirical distribution of $r(s)$ at the assumed default time s.

Recall that in this model all zero-coupon bond values at s are functions of $r(s)$. For $T_i > s$ we write

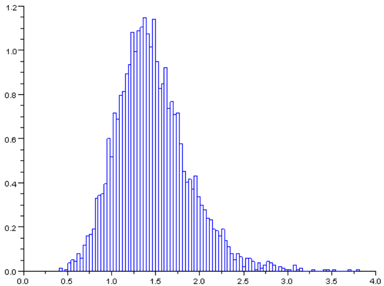
$$p_i(s, r(s)) = p(s, T_i)(r(s)).$$

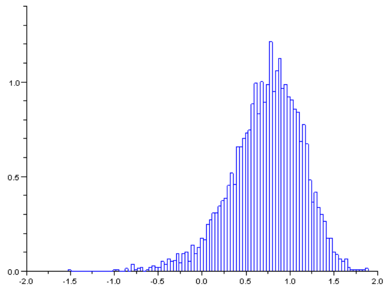
Example: $s = 2.5$ years in a 5-year swap.

$\rho = 0$:



$\rho = -0.6, +0.6$





Risk measures

To a close approximation (exact at coupon dates) the swap value at time s is

$$V_s = V_s(r(s)) = 1 - p_n(s, r(s)) - K \sum_{i \geq k(s)} \delta_i p_i(s, r(s))$$

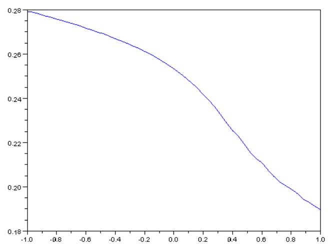
The *Expected Positive Exposure* is

$$\mathrm{EPE}_s = \mathbb{E}[V_s^+ | \tau_0 = s].$$

The *Credit Value Adjustment* is the value of compensation for losses on default, i.e.

$$\mathrm{CVA} = \mathbb{E}\left[e^{-\int_0^{\tau_0 \wedge T_n} r(u)du} V^+_{\tau_0 \wedge T_n} \right]$$

We can evaluate these by Monte Carlo simulation. CVA as function of ρ:



Concluding Remarks

We have developed a joint model for asset values and counterparty default risk, which enables us to estimate counterparty risk.

However there is lots more to do:

- More efficient computational methods.
- Multi-asset problems.
- Inclusion of credit assets (CDOs,...)
- And the big one: how to get a consistent procedure for calibrating ρ.

Appendix: A general (but maybe useless) representation

Suppose we have a general short rate model $r(t) = r(t, X_t)$ where X_t is a multidimensional diffusion process driven by BM $W \in \mathbb{R}^{n+1}$. As before we use barrier crossing by one component of W, or some linear combination, as default indicator. Then we can represent X_t as the solution of an SDE in the form

$$df(X_t) = L_0 f(X_t)dt + Zf(X_t) \circ dB_t^0 + L_j f(X_t) \circ dB_t^j \tag{15}$$

where $L_0, \ldots, L_n, Z$ are vector fields, $B^0, \ldots, B^n$ are independent BM and '$\circ$' denotes the Stratonovich integral.

We want to replace B^0 by a Bessel bridge, say Y.

We can use ideas of Doss, Sussman, Kunita to obtain a conditional representation of X_t given a sample path of B^0.

Let $\zeta_t(x) = \zeta(t,x)$ denote the flow of the vector field Z, i.e. the unique solution of the equation

$$\begin{aligned} \frac{d}{dt} f(\zeta_t(x)) &= Z f(\zeta_t(x)), \quad f \in C^\infty(\mathbb{S}) \\ \zeta_0(x) &= x. \end{aligned}$$

This is a diffeomorphism for each $t \geq 0$. Define

$$\xi_t(x) = \zeta_{Y(t)}(x).$$

As is easily checked, $\xi = \xi_t(x)$ is the solution of

$$d\xi_t = Z(\xi_t) \circ dY_t$$

and $\xi_t(\cdot)$ is almost surely a diffeomorphism for each $t > 0$.

Now consider the equation

$$df(\eta_t) = \xi_{t*}^{-1} L_0 f(\eta_t) dt + \xi_{t*}^{-1} L_j f(\eta_t) \circ dB_t^j. \tag{16}$$

This equation has a unique solution and it follows by applying the Ito formula that

$$\begin{aligned} X_t(x) &= \xi_t \circ \eta_t(x) \\ &= \zeta(Y(t), \eta_t(x)). \end{aligned} \tag{17}$$

The representation (16), (17) describes the behaviour of X_t conditioned on Y. Recall that the map ξ_{t*}^{-1} is parametrized by Y and that $B^0, B^j, j = 1, 2..$ are independent BM. Thus, conditional on B^0, η_t is a diffusion process whose differential generator is

$$A_t^* = \xi_{t*}^{-1} L_0 + \sum_j (\xi_{t*}^{-1} L_j)^2$$

and, for each $t > 0$, X_t is diffeomorphically related to η_t by equation (17).