Good Deal Bounds

Tomas Björk,
Department of Finance,
Stockholm School of Economics,

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Outline

- Overview of the general theory of GDB
  (Irina Slinko & Tomas Björk)

- Applications to vulnerable options
  (Agatha Murgoci)

- Applications to regime switching models
  (Catherine Donnelly)

1. General theory

- Cochrane, J., and Saá Requejo, J. "Beyond arbitrage: Good-deal asset price bounds in incomplete markets". *Journal of Political Economy 108* (2000), 79–119.

- Björk, T., and Slinko, I. "Towards a general theory of good deal bounds". *Review of Finance 10*, (2006), 221-260.

Basic Framework

Exogenously Given:

- An underlying **incomplete** market.
- A contingent T-claim Z.

Recall: The arbitrage free price of Z is given by

$$\Pi(t, Z) = E^P\left[\frac{D_T}{D_t} \cdot Z \middle| \mathcal{F}_t\right] = E^Q\left[e^{-\int_t^T r_u du} \cdot Z \middle| \mathcal{F}_t\right]$$

where D is the stochastic discount factor (SDF)

$$D_t = e^{-\int_0^t r_u du} L_t, \quad L_t = \frac{dQ}{dP}, \quad \text{on } \mathcal{F}_t$$

However:

- Incomplete market $\Rightarrow$ D and Q are not unique.
- Thus no unique price process $\Pi(t, Z)$.

How can we price in this incomplete setting?

Sad Fact:
The no arbitrage bounds are far to wide to be useful.

Some standard techniques:

- Quadratic hedging.
- Utility indifference pricing.
- Minimize some distance between Q and P.

Our Goal:

- Find "reasonable" and **tight** no arbitrage bounds.
- Economic interpretation.
- Market data as input.

Cochrane and Saa-Requejo

- An arbitrage opportunity is a "ridiculously good deal".

- Thus, no arbitrage pricing is pricing subject to the constraint of ruling out ridiculously good deals.

The CSR Idea:
Find pricing bounds by ruling out, not only ridiculously good deals, but also "unreasonably good deals".

How is this formalized?:

- Impose restrictions on the volatility of the SDF (stochastic discount factor).

- Impose bounds on the Sharpe Ratio!

Sharpe Ratio

The Sharpe Ratio for an asset price S is defined by

$$SR = \text{risk premium per unit volatility}$$

i.e.

$$SR = \frac{\mu - r}{v}$$

where

$$\begin{aligned} \mu &= \text{mean rate of return} \\ r &= \text{short rate} \\ v &= \text{total volatility of } S \end{aligned}$$

i.e.

$$v_t^2 dt = Var^P \left[\left. \frac{dS_t}{S_{t-}} \right| \mathcal{F}_{t-} \right]$$

Moral:
High Sharpe Ratio = unreasonbly good deal.

Reasonable Values of the Sharp Ratio

- The market portfolio is not so dramatically inefficient $\Rightarrow$ we do not expect to see SR much higher then historical market SR, which is about 0,5.

- Using utility function approach, unless we make extreme assumptions about consumption volatility and risk aversion it is difficult to generate SR higher then 0,3.

- A hedge fund with a SR around 2 is doing extremely well.

CSR First Problem Formulation

Find upper and lower price bounds subject to a constraint of the Sharpe Ratio, i.e. find

$$\sup \; E^P \left[\frac{D_T}{D_t} \cdot Z \middle| \mathcal{F}_t \right]$$

subject to

$$|SR_t| \leq B. \quad \text{for all } t$$

However:

- Formulated this way, the problem is mathematically intractable.

- Even if we have a bound on the SR for the Z derivative, it may be possible to form portfolios (on underklying and derivative) with very high Sharpe ratios.

Reformulating the Constraint

Recall:
In a Wiener driven world we have the

Hansen-Jagannathan inequality:

$$|SR_t|^2 \leq \|h_t\|^2_{R^d}$$

where

$$-h_t = \text{market price vector of } W\text{-risk}$$

or in martingale language

$$dL_t = L_t h_t dW_t, \quad L_t = \frac{dQ}{dP}, \quad \text{on } \mathcal{F}_t$$

Idea:
Replace SR constraint with constraint on $\|h_t\|$

Second CSR Problem Formulation

Find

$$\sup_h \; E^P\left[\frac{D_T}{D_t}\cdot Z \middle| \mathcal{F}_t\right]$$

subject to

$$\|h_t\|^2_{R^d} \le B^2 \quad \forall t \in [0,T].$$

CSR Results:

- Main analysis done in one-period framework.
- In continuous time, CSR derive a PDE for upper and lower price bounds through (informal) dynamic programming argument.
- Obtains nice numerical results.
- Surprisingly tight bounds.

Limitations of CSR

$$\sup_h \; E^P\left[\frac{D_T}{D_t}\cdot Z \,\middle|\, \mathcal{F}_t\right]$$

subject to

$$\|h_t\|^2_{R^d} \leq B^2 \quad \forall t \in [0,T].$$

- Only Wiener driven asset price processes.
- Analysis carried out entirely in terms of SDFs.
- Connection to martingale measures not clarified.
- CSR derive a HJB equation, but the precise underlying control problem is never made precise.
- Some ad hoc assumptions on the upper an lower bounds processes.

Main Contributions of the Present Paper

- We focus on martingale measures rather than on SDF, which is mathematically equivalent but
  - allows to use the technical machinery of martingale theory
  - considerably streamlines the arguments - "good-deal" pricing problem can be formulated as a **standard stochastic control problem**

- We **do not** assume the existence, **nor do we** make assumptions about the explicit dynamics of the price bounds

- We introduce a driving general marked **point process**, thus allowing the possibility of jumps in the random processes describing the financial markets.

A Generic Example

The Merton model:

$$dS_t = S_t \alpha dt + S_t \sigma dW_t + S_{t-} \delta_t dN_t$$

Here N is Poisson and δ lognormal at jumps.

- To obtain a unique derivatives pricing formula Merton assumes **zero market price of jump risk**.

Can we do better?

The Model

- An n-dimensional traded asset price process $S = (S^1, \ldots, S^n)$

$$
\begin{aligned}
dS_t^i &= S_t^i \alpha_i (S_t, Y_t)\, dt + S_t^i \sigma_i (S_t, Y_t) dW_t \\
&\quad + S_{t-}^i \int_X \delta_i (S_{t-}, Y_{t-}, x) \mu(dt, dx), \quad i = 1, \ldots, n
\end{aligned}
$$

- A k-dimensional factor process $Y = (Y^1, \ldots, Y^n)$

$$
\begin{aligned}
dY_t^j &= a_j (S_t, Y_t)\, dt + b_j (S_t, Y_t) dW_t \\
&\quad + \int_X c_j (S_{t-}, Y_{t-}, x) \mu(dt, dx). \quad j = 1, \ldots, k
\end{aligned}
$$

Recap on Marked Point Processes

- $\mu(dt, dx)$ - number of events in $(dt, dx) \in R_+ \times X$

- Typically we assume that $\mu(dt, dx)$ has predictable P-intensity measure process λ This essentially means that

$$\lambda_t(dx)dt = E^P\left[\mu(dt, dx) \middle| F_{t-}\right]$$

- $\lambda_t(dx)$- expected rate of events at time t with marks in dx.

- For each x, the differential $\mu(dt, dx) - \lambda_t(dx)dt$ is a P-martingale differential.

- $\lambda_t(X)$=global intensity (regardless of mark)

- The probability distribution of marks, given that there is a jump at t is

$$\frac{1}{\lambda_t(X)} \cdot \lambda_t(dx)$$

Assumptions

- The point process μ has a predictable P-intensity measure λ, of the form

$$\lambda_t(dx) = \lambda(S_{t-}, Y_{t-}, dx)dt.$$

- We assume the existence of a short rate r of the form

$$r_t = r(S_t, Y_t).$$

- We assume that the model is free of arbitrage in the sense that there exists a (not necessarily unique) risk neutral martingale measure Q.

- $\delta_i(s, y, x) \geq -1 \quad \forall i \quad$ and $\quad \forall (s, y, x)$

- We consider claims of the form

$$Z = \Phi(S_T, Y_T)$$

Girsanov for MPP and Wiener

Assume that $\mu(dt, dx)$ has predictable P-intensity $\lambda_t(dx)$ and that W is d-dimensional P-Wiener

- Choose predictable processes h_t and $\varphi_t(x) \geq -1$
- Define likelihood process L by

$$\begin{cases} dL_t &= L_t h_t dW_t + L_{t-} \int_X \varphi_t(x) \tilde{\mu}(dt, dx) \\ L_0 &= 1 \end{cases}$$

$$\tilde{\mu}(dt, dx) = \mu(dt, dx) - \lambda_t(dx)dt$$

Then:

- $\mu(dt, dx)$ has Q-intensity

$$\lambda_t^Q(dx) = \{1 + \varphi_t(x)\}\, \lambda_t(dx)$$

- We have

$$dW = h_t^\star + dW_t^Q$$

Extended Hansen-Jagannathan Bounds

Proposition:
For all arbitrage free price processes S and for all Girsanov kernels $h_t, \varphi_t(x)$, defining a martingale measure, the following inequality holds

$$|SR_t|^2 \leq \|h_t\|_{R^d}^2 + \int_X \varphi_t^2(x)\lambda_t(dx)$$

or

$$|SR_t|^2 \leq \|h_t\|_{R^d}^2 + \|\varphi_t\|_{\lambda_t}^2,$$

where $\|\cdot\|_{\lambda_t}$ denotes the norm in the Hilbert space $L^2\left[X, \lambda_t(dx)\right]$.

Good Deal Bounds

The upper good deal price bound process is defined as the optimal value process for the following optimal control problem.

$$V(t,s,y) = \sup_{h,\varphi} \quad E^Q\left[e^{-\int_t^T r_u du} \Phi\left(S_T, Y_T\right) \Big| \mathcal{F}_t \right]$$

Q dynamics:

$$\begin{aligned} dS_t^i &= S_t^i \left\{ r_t - \int_X \delta_i(x)\left\{1+\varphi_t(x)\right\}\lambda_t(dx) \right\} dt \\ &\quad + S_t^i \sigma_i dW_t^Q + S_{t-}^i \int_X \delta_i(x)\mu(dt,dx), \\ &\quad i = 1,\ldots,n \end{aligned}$$

$$\begin{aligned} dY_t^j &= \left\{a_j + b_j h_t\right\} dt + b_j dW_t^Q \\ &\quad + \int_X c_j(x)\mu(dt,dx). \quad j = 1,\ldots,k \end{aligned}$$

Standard stochastic control problem

Constraints on h and φ

- (Guarantees that Q is a martingale measure)

$$\alpha_i + \sigma_i h_t + \int_X \delta_i(x)\left\{1+\varphi_t(x)\right\}\lambda_t(dx) = r_t, \quad \forall i$$

- (Rules out "good deals")

$$\|h_t\|^2_{R^d} + \int_X \varphi_t^2(x)\lambda_t(dx) \leq B^2,$$

- (Ensures that Q is a positive measure)

$$\varphi_t(x) \geq -1, \quad \forall t, x.$$

HJB Equation

Theorem The upper good deal bound function is the solution V to the following boundary value problem

$$
\begin{aligned}
\frac{\partial V}{\partial t}(t,s,y) + \sup_{h,\varphi} A^{h,\varphi}V(t,s,y) - r(s,y)V(t,s,y) &= 0, \\
V(T,s,y) &= \Phi(s,y)
\end{aligned}
$$

NB:
The embedded static problem

$$\sup_{h,\varphi}\left\{A^{h,\varphi}V(t,s,y)\right\}$$

is a full fledged variational problem. For each (t,s,y) we have to determine $\varphi(t,s,y,\cdot)$ as a function of x.

$$
\begin{aligned}
& A^{h,\varphi} V(t,s,y) \\
= & \sum_{i=1}^{n} \frac{\partial V}{\partial s_i} s_i \left\{ r - \int_X \delta_i(x) \{1 + \varphi(x)\} \lambda_t(dx) \right\} \\
+ & \sum_{j=1}^{k} \frac{\partial V}{\partial y_j} \{a_j + b_j h\} + \int_X \Delta V(x) \{1 + \varphi(x)\} \lambda_t(dx) \\
+ & \frac{1}{2} \sum_{i,l=1}^{n} \frac{\partial^2 V}{\partial s_i \partial s_l} s_i s_l \sigma_i^{\star} \sigma_l + \frac{1}{2} \sum_{j,l=1}^{k} \frac{\partial^2 V}{\partial y_j \partial y_l} b_j^{\star} b_l + \sum_{i,j=1}^{k} \frac{\partial^2 V}{\partial s_i \partial y_j} s_i \sigma_i^{\star} b_j
\end{aligned}
$$

Here

$$
\Delta V(x) = V\left(t, s(1+\delta(x)), y + c(x)\right) - V(t,s,y)
$$

Example: The Compound Poisson-Wiener Model

Consider a financial market and a scalar price process S satisfying the SDE

$$dS_t = S_t \alpha dt + S_t \sigma dW_t + S_{t-} \int_X \delta(x) \mu(dt, dx).$$

The point process μ has a P-compensator of the form

$$\nu^P(dt, dx) = \lambda(dx) dt$$

λ is a finite nonnegative measure on $(X, \mathcal{X})$.

In this case the static problem has the following form

$$\max_{h,\varphi} \quad \int_X \Delta V(t,s,x)\varphi(t,s,x)\lambda(dx) \\ -sV_s(t,s)\int_X \delta(x)\varphi(t,s,x)\lambda(dx),$$

subject to

$$\begin{aligned} \alpha + \sigma h + \int_X \delta(x)\lambda(dx) + \int_X \delta(x)\varphi(x)\lambda(dx) &= r, \\ h^2 + \int_X \varphi^2(x)\lambda(dx) &\leq B^2, \\ \varphi(x) &\geq -1, \end{aligned}$$

where, as before,

$$\Delta V(t,s,x) = V\left(t, s\left[1+\delta(x)\right]\right) - V(t,s).$$

- The static problem has to be solved for every fixed choice of (t, s, y) and the control variables are h and φ

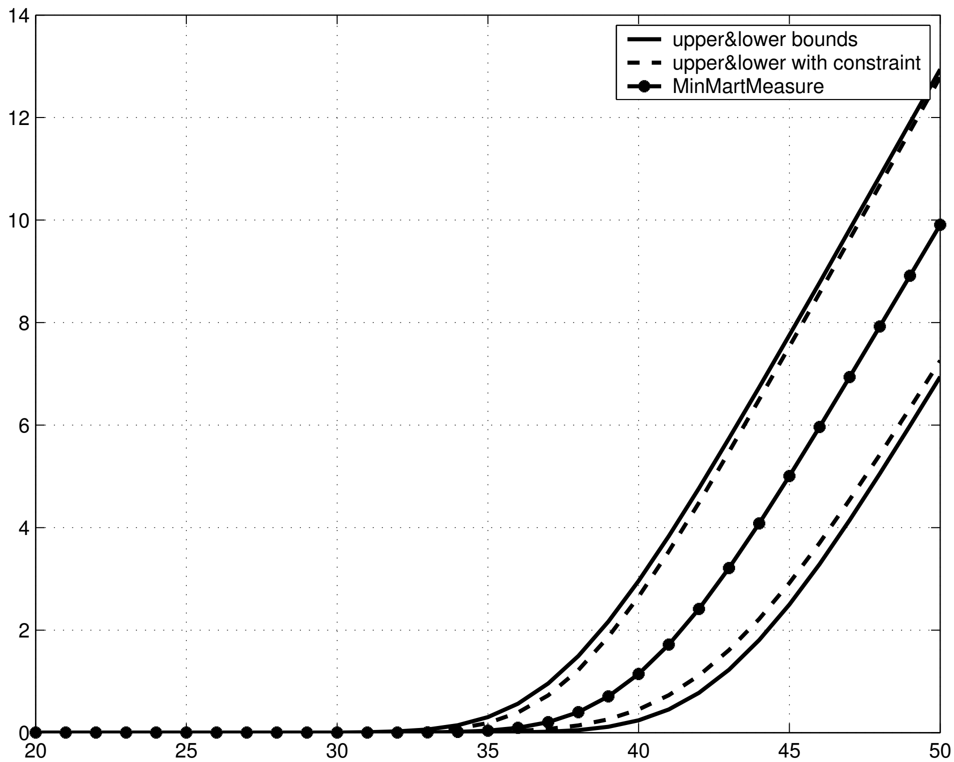
- For fixed (t, s, y) h is d-dimensional vector

- However, φ is a function of x and thus infinite-dimensional control variable

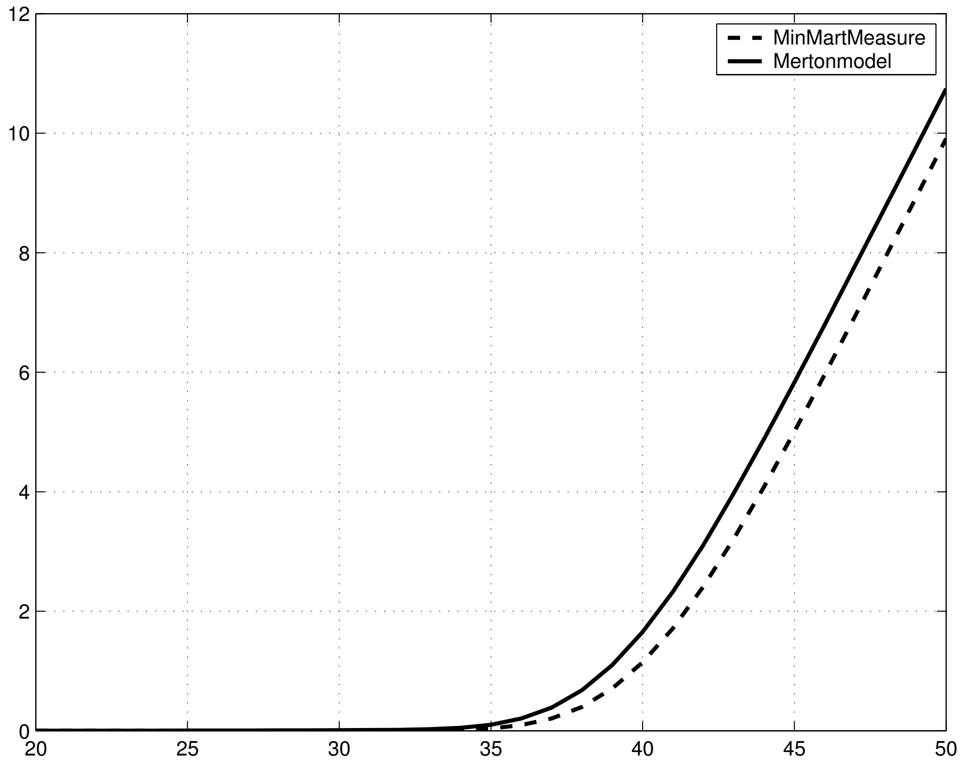
- We are thus facing a variational problem inside the HJB equation.

- We have to resort to numerical methods.

Good deal pricing bounds



The minimal martingale measure and the Merton model



Taylor Approximation

Disturbing Fact:
The bounds are computationally demanding.

Idea:
Write the upper bounds as $V(t, s, B)$ and make a Taylor expansion in B around B_0, corresponding to the MMM.

$$V(t, s, B) = V(t, s, B_0) + (B - B_0)\frac{\partial V}{\partial B}(t, s, B_0)(t, s, B_0)$$

However:

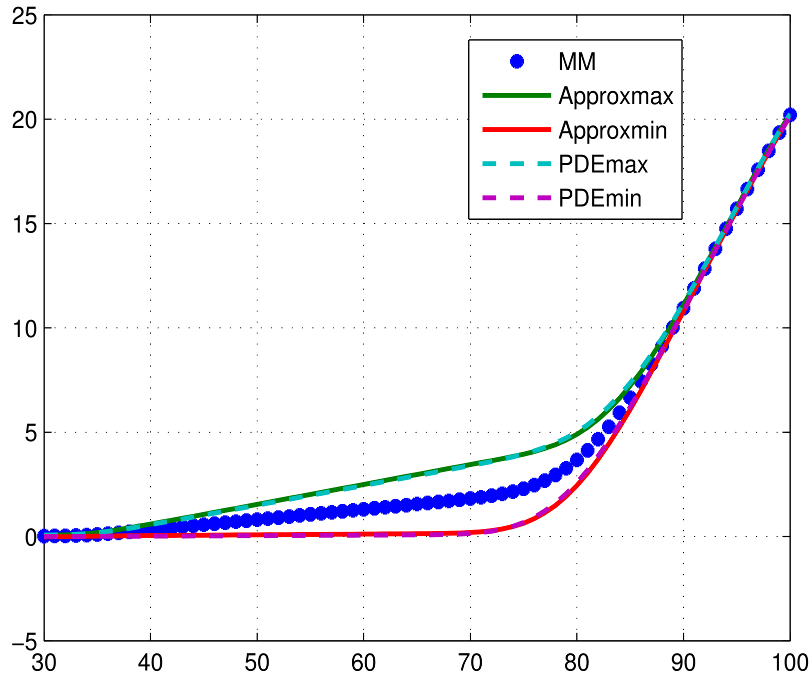
$$\frac{\partial V}{\partial B}(t, s, B_0) = +\infty$$

Modified idea:
Do the expansion in the rescaled variable

$$\sqrt{B^2 Var_P[\frac{dS}{S}] - R^2}$$

where R is the excess rate of return.

Example: Wiener-Poisson



Ongoing work...

2. Vulnerable options

Murgoci, A. "Vulnerable Options and Good Deal Bounds - A Structural Model". Working paper. Copenhagen Business School.

Murgoci, A. "Pricing Counter-Party Risk Using Good Deal Bounds". Working paper. Copenhagen Business School.

Counter-party Risk

- Brought to the forefront by recent events
- Partly due to trading on OTC markets

Model

- Traded stock S, with dynamics

$$dS_t = \alpha_t S_t dt + S_t \gamma_t d\tilde{W}_t^P,$$

- Bank account with dynamics

$$dB_t = rB_t dt$$

- Default indicator Y.

Assumption. We assume that Y is a a counting process. Two cases are considered.

- Constant intensity

- Stochastic intensity λ_t where

$$d\lambda_t = \kappa(\theta - \lambda_t)dt + \sigma\sqrt{\lambda_t}dW_t^P$$

The Payoff Function

- Vulnerable European call

$$X = \begin{cases} \max[S_T - K, 0], & \text{if} \quad Y_T = 0, \\ \mathcal{R}, & \text{if} \quad Y_t > 0, \text{ for some } 0 < t \leq T \end{cases}$$

The martingale measure Q

- Dynamics for the Radon-Nikodym derivative $L = dQ/dP$

$$
\begin{aligned}
dL_t &= L_t h_t d\tilde{W}_t^P + L_t g_t \sqrt{\lambda} dW_t^P + L_{t-}\varphi_t (dN_t - \lambda_t dt) \\
L_0 &= 1
\end{aligned}
$$

- Positivity constraint: $\varphi_t \geq -1$
- Martingale constraint: $r = \alpha_t + \gamma_t h_t$
- Good deal bound constraint

$$
h_t^2 + g_t^2 \lambda + \varphi_t^2 \lambda_t \leq C^2
$$

The Lower Good Deal Bound Price

Optimal control problem:

$$\min_{h,g,\varphi} \quad E^Q\left[e^{-r(T-t)+\int_t^T q\lambda_u^Q du}\cdot\Phi(S_T)\middle|\mathcal{F}_t\right]$$

$$dS_t = rS_t dt + S_t\gamma_t d\tilde{W}_t$$

$$d\lambda_t = \kappa\left(\theta - \lambda_t + g_t\sigma\lambda_t\right)dt + \sigma\sqrt{\lambda_t}dW_t$$

$$\lambda_t^Q = \lambda_t(1+\varphi_t)$$

$$\alpha_t + \gamma_t h_t = r$$

$$\varphi_t \geq -1$$

$$h_t^2 + g_t^2\lambda + \varphi_t^2\lambda_t \leq C^2$$

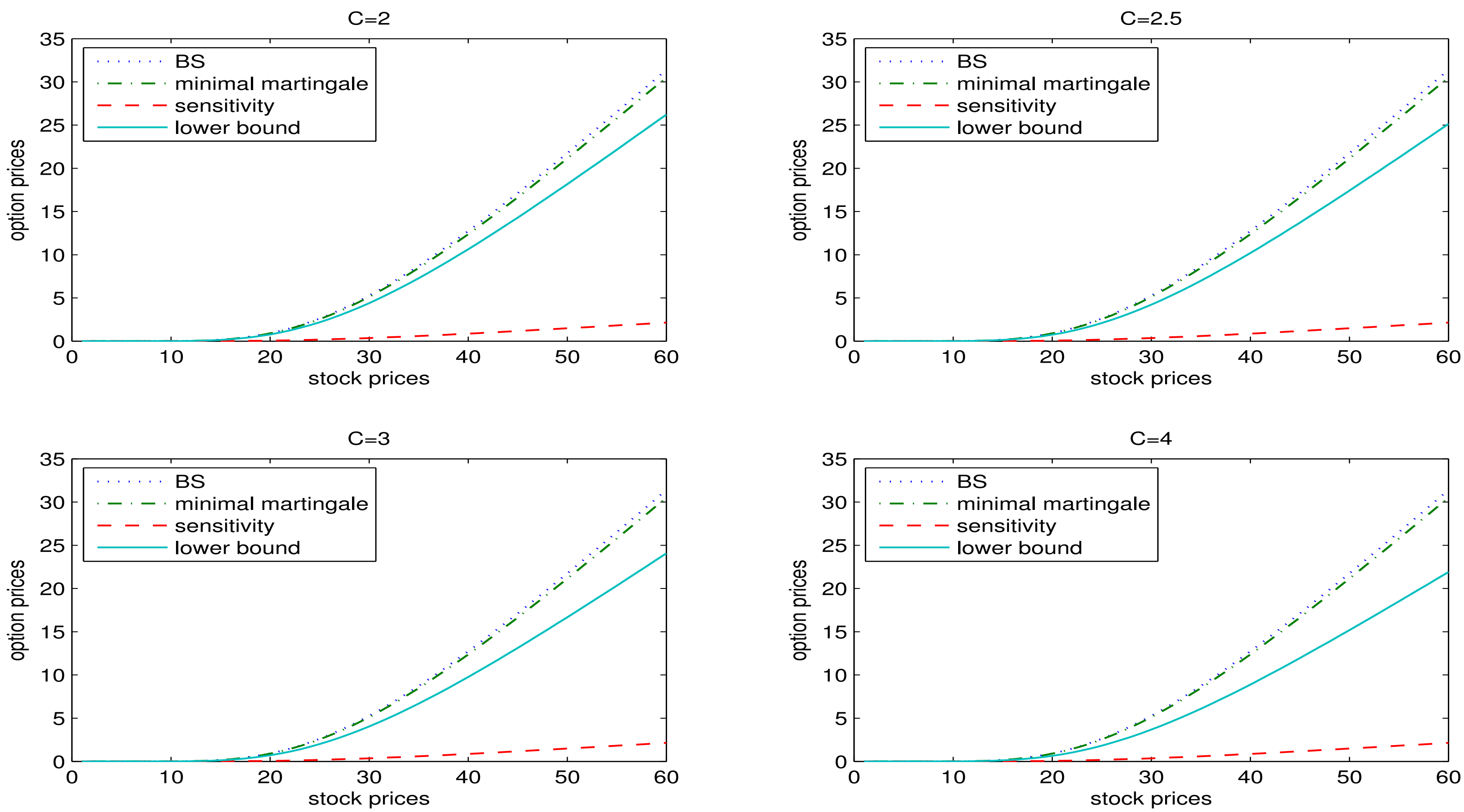
Hamilton Jacobi Bellman Equation

$$
\begin{aligned}
\frac{\partial V}{\partial t}(t,s,y,\lambda) + \inf_{h,g,\varphi} \mathcal{A}^{h,g,\varphi} V(t,s,y,\lambda) - rV(t,s,y,\lambda) &= 0 \\
V(T,s,0,\lambda) &= \max[S_T - K, 0] \\
V(t,s,1,\lambda) &= \mathcal{R}
\end{aligned}
$$

- Solving for each t, s, y, λ the embedded static problem
  $\longrightarrow$ we obtain the Girsanov Kernel

- Solving the PDE
  $\longrightarrow$ we obtain the price of the vulnerable option

GDB for different GDB constraints



3. Regime switching models

DONNELLY, C. "Good-deal bounds in a regime switching market". Working paper. ETH, Zurich.

Model

Random sources:

$$
\begin{aligned}
W_t &= \text{Standard Wiener process} \\
\alpha_t &= \text{Continuous time Markov chain on } \{1, 2, \ldots, d\} \\
G &= \text{Intensity matrix for } \alpha
\end{aligned}
$$

.

Price dynamics:

$$
\begin{aligned}
dS_t &= S_t \mu(\alpha_t) dt + S_t \sigma(\alpha_t) dW_t, \\
dB_t &= r(\alpha_t) B_t dt
\end{aligned}
$$

Claim to be priced:

$$
X = \Phi(S_T, \alpha_T)
$$

Highly incomplete market

Girsanov for α

We define the counting process N^{ij} by

$$N_t^{ij} = \sum_{0 \le s \le t} \mathbf{I}\left\{\alpha_{s-} = i,\ \alpha_s = j\right\}, \quad i \ne j$$

Intensity process for N_t^{ij}

$$\lambda_t^{ij} = g_{ij}\mathbf{I}\left\{\alpha_{t-} = i\right\}$$

Corresponding martingale:

$$M_t^{ij} = N_t^{ij} - \int_0^t \lambda_s^{ij} ds$$

Girsanov Theorem for α

- Define L by

$$\begin{cases} dL_t &= L_{t-}h_t dW_t + L_{t-}\sum_{i\neq j}\varphi_t^{ij}dM_t^{ij} \\ L_0 &= 1 \end{cases}$$

  where $\varphi^{ij} > -1$.

- Define Q by $L_t = \frac{dQ}{dP}$, on $\mathcal{F}_t$

Then:

- The intensity of N^{ij} under Q is given by

$$\tilde{\lambda}_t^{ij} = \lambda_t^{ij}\left(1 + \varphi_t^{ij}\right)$$

- We have

$$dW_t = h_t dt + dW_t^Q$$

  where W^Q is Q-Wiener.

Admissible kernels

A Girsanov kernel (h, φ) is **admissible** if Q is a martingale measure for all traded assets, underlying and derivative, in the market.

Hansen-Jagannathan: For every admissible Girsanov kernel process (h, φ) and for every asset in the market we have

$$(SR)_t^2 \leq h_t^2 + \sum_{i \neq j} |\varphi_t^{ij}|^2 \lambda_t^{ij}.$$

Martingale condition

Recall price dynamics

$$dS_t = S_t \mu_t dt + S_t \sigma_t dW_t$$

A Girsanov kernel (h, φ) satisfies the martingale condition iff

$$\mu_t + \sigma_t h_t = r_t$$

The Girsanov kernel h is uniquely determined, but we have no restriction on φ.

The GDB problem

For a contingent claim $Z = \Phi(S(T), \alpha(T))$, the upper good deal price process V is the optimal value process for the control problem

$$\sup_{h,\varphi} E^Q \left[e^{-\int_t^T r_s ds} \Phi(S_T, \alpha_T) \middle| \mathcal{F}_t \right]$$

where the predictable processes (h, φ) are subject to the constraints

$$\begin{aligned} h_t &= \frac{r_t - \mu_t}{\sigma_t}, \\ \varphi_t^{ij} &\geq -1, \end{aligned}$$

$$h_t^2 + \sum_{i \neq j} |\varphi_t^{ij}|^2 \lambda_t^{ij} \leq B^2.$$

The HJB eqn

Assume

$$h_t = h(t, S_t, \alpha_{t-}) \quad \varphi_t^{ij} = \varphi^{ij}(t, S_t, \alpha_{t-})$$

The HJB eqn for the optimal value function V is given by

$$\begin{aligned} \frac{\partial V}{\partial t}(t,x,i) + \sup_{(h,\varphi)} \mathbf{A}^{h,\varphi} V(t,x,i) - r(t,x,i)V(t,x,i) &= 0 \\ V(T,x,i) &= \Phi(x,i) \end{aligned}$$

System of PIDEs.

The infinitesimal operator

$$
\begin{aligned}
\mathbf{A}^{h,\varphi} V(t,x,i) \quad &= r(t,x,i) x V_x(t,x,i) + \tfrac{1}{2}\sigma^2(t,x,i) x^2 V_{xx}(t,x,i) \\
&\quad + \textstyle\sum_j g_{ij} \left(1 + \varphi_t^{ij}\right) \{V(t,x,j) - V(t,x,i)\}
\end{aligned}
$$

Numerical Example

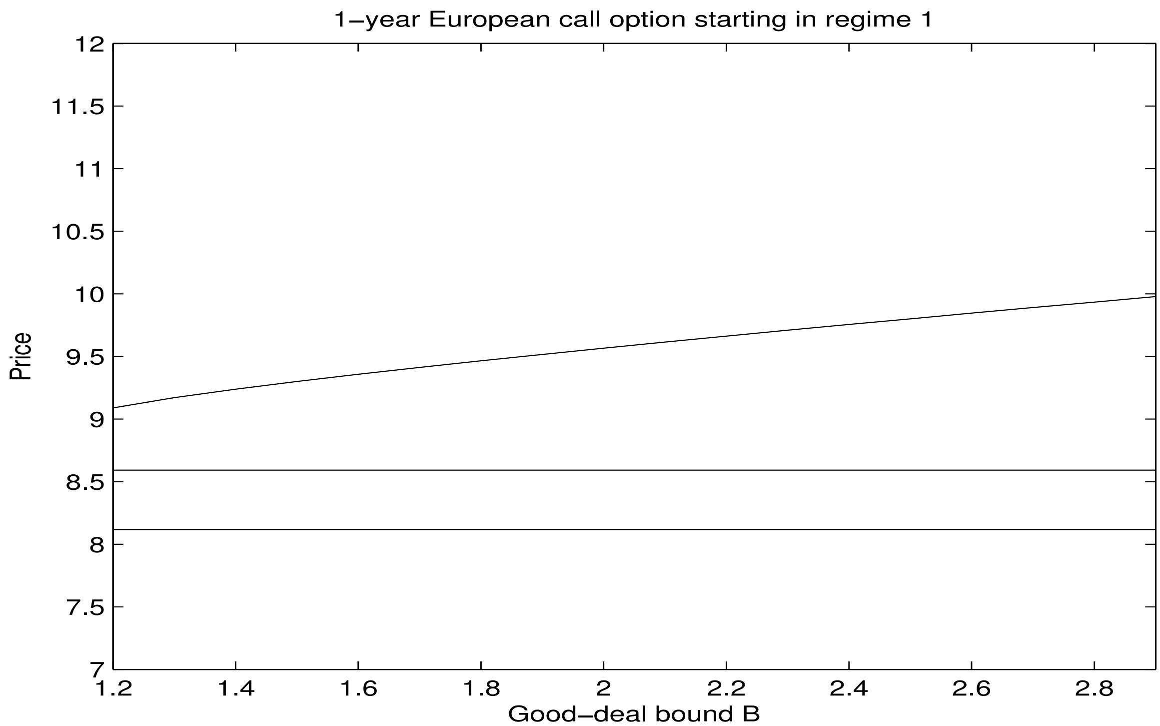
Regime switching models with two regimes.

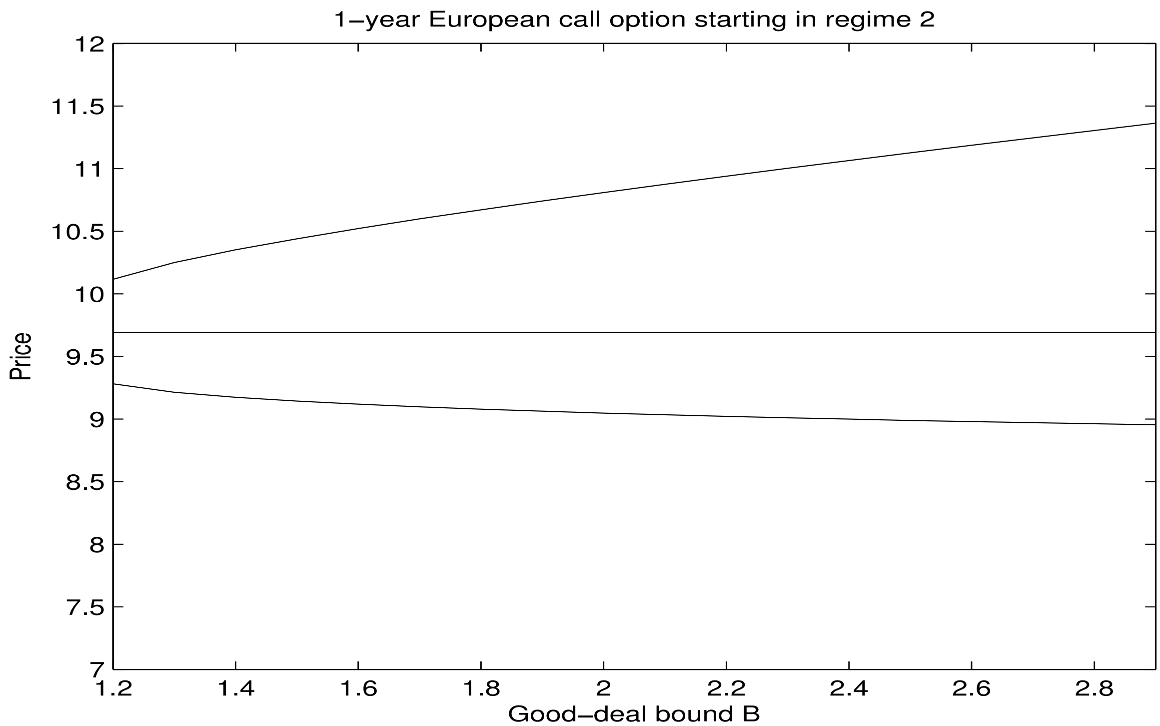
Market parameters based on Hardy (2001):

i	$r(i)$	$\mu(i)$	$\sigma(i)$
1	0.06	0.15	0.12
2	0.06	-0.22	0.26

Generator

$$G = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} -0.5 & 0.5 \\ 5 & -5 \end{pmatrix}.$$


1−year European call option starting in regime 1
12
11.5
11
10.5
10
9.5
9
8.5
8
7.5
7
Price
1.2
1.4
1.6
1.8
2
2.2
2.4
2.6
2.8
Good−deal bound B


1−year European call option starting in regime 2
Price
Good−deal bound B
12
11.5
11
10.5
10
9.5
9
8.5
8
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7
1.2
1.4
1.6
1.8
2
2.2
2.4
2.6
2.8

Further research areas

- GDB pricing for credit risk models where the credit rating evolves as a Markov chain. This would be an interesting application of Donnelly's technique.

- Does there exist a theory for good deal hedging?